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Zero cancellation for general rational matrix functions ☆

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ABSTRACT

The problem of cancelling a specified part of the zeros of a completely general rational matrix function by multiplication with an appropriate invertible rational matrix function is investigated from different standpoints. Firstly, the class of all factors that dislocate the zeros and feature minimal McMillan degree are derived. Further, necessary and sufficient existence conditions together with the construction of solutions are given when the factor fulfills additional assumptions like being J-unitary, or J-inner, either with respect to the imaginary axis or to the unit circle. The main technical tool are centered realizations that deliver a sufficiently general conceptual support to cope with rational matrix functions which may be polynomial, proper or improper, rank deficient, with arbitrary poles and zeros including at infinity. A particular attention is paid to the numerically-sound construction of solutions by employing at each stage unitary transformations, reliable numerical algorithms for eigenvalue assignment and efficient Lyapunov equation solvers.

1. Introduction

A general rational matrix function (rmf) is characterized by its structural elements: finite and infinite poles and zeros, together with their partial multiplicities (as defined by the Smith–McMillan form [23], see also [8,9]), and the minimal indices of a polynomial basis of the null space to the left and right (as defined by Forney [10]).

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An interesting problem with many applications in linear systems theory is to eliminate (cancel or dislocate) some of the structural elements of a rmf $R(\lambda)$ by multiplication with an invertible rmf $R_\ell(\lambda)$, i.e., $R_\ell(\lambda)R(\lambda) = \widehat{R}(\lambda)$, where $\widehat{R}(\lambda)$ has only part of the structural elements of the original $R(\lambda)$.

Since a multiplication to the left with an invertible factor $R_\ell(\lambda)$ does not change the right null space, we can alter with $R_\ell(\lambda)$ only poles, zeros and left minimal indices. Dually, by multiplication to the right with an invertible rmf we can change poles, zeros and/or right minimal indices. The problems to the left and right are dual to one another and therefore we will consider further only those to the left. As the solutions to these structural displacement problems are in general highly nonunique, one usually seeks solutions featuring minimum McMillan degree. We call such solutions *minimal*.

The problem of eliminating the minimal indices has been recently solved in [27] where the class of solutions has been characterized for a completely general rational matrix function. In [27] several solutions are given: minimal, J-unitary and J-inner, either with respect to the imaginary axis or the unit circle.

The problems of cancelling part of the poles or zeros have a longer history. They have been originally considered in [4] and more elaborately in [7,32] where it has been shown that it is always possible to find a nonsingular $R_\ell(\lambda)$ of McMillan degree 1 (having one pole and one zero) such that in the product $R_c(\lambda) = R_\ell(\lambda)R(\lambda)$ the zero of $R_\ell(\lambda)$ cancels a pole of $R(\lambda)$ or the pole of $R_\ell(\lambda)$ cancels a zero of $R(\lambda)$ (or both). Since there is a certain degree of freedom in choosing the invertible factor, one can add some supplementary conditions on it, like for example to be unitary. The solutions in [7,32] based on a transfer function approach have been refined and streamlined for finite poles and zeros in [35] by using state-space realizations. In [35] necessary and sufficient existence conditions and the construction of solutions are given either by using an one-shot approach in which the simultaneous cancellation of all undesirable poles (zeros) is performed at once or by a recursive scheme in which the poles (zeros) are cancelled one by one. The case in which $R_\ell(\lambda)$ is required to be unitary with respect to the imaginary axis or the unit circle is included for the recursive approach only. An alternative approach for poles cancellation with J-inner factors may be found in Chapter 5 of [22].

The pole cancellation problem for a general rational matrix has been fully investigated in [25], where necessary and sufficient solvability conditions for general or J-unitary factors (either on the imaginary axis or the unit circle) are given together with the construction of the class of minimal solutions, both in the canonical and noncanonical cases. More recently, and apparently unaware of [25], a characterization of the pole displacement factors that are J-unitary with respect to the imaginary axis is given in [8] for a proper rmf along with other interesting characterizations.

The elimination of zeros by J-unitary (or J-inner) factors is considered in [9] by reduction to the pole cancellation case. However, only the case of a proper rmf $R(\lambda)$ is considered and either necessary or sufficient solvability conditions are provided in the case in which $R(\lambda)$ is not of full column rank. The cancellation of infinite zeros of a proper non-square rmf of full rank is considered in [38] with the motivation of constructing an interactor.

The primary goal of this paper is to extend the results on zero cancellation considered in [9,35,38], in the following directions. First, we allow the given rmf to be completely general, possibly improper or polynomial, and without any restriction on its structural elements. Further, we characterize the class of minimal McMillan degree invertible factors that are able to cancel any desired zeros of $R(\lambda)$, possibly including those at infinity. Finally, we consider J-symmetries with respect to both the imaginary axis and the unit circle, and give solvability conditions that are simultaneously necessary and sufficient. All solutions we propose are based as much as possible on constant unitary transformations and numerically-sound solutions of pencil equations and eigenvalue assignment problems gaining the benefit of numerical reliability of the underlying algorithms. Specifically, we deal with the following problem.

Zero Displacement Problem (**ZDP**). Let $R(\lambda)$ be a general rmf and $\Gamma_g \cup \Gamma_b$ a disjoint partition of the closed complex plane. Find the class of invertible rmf's $R_\ell(\lambda)$ of mimimal McMillan degree with zeros in Γ_g such that all the zeros of

$$\widehat{R}(\lambda) := R_\ell(\lambda)R(\lambda) \tag{1}$$

also lie in Γ_g.

Remark 1. The solution $R_\ell(\lambda)$ is required to have zeroes in Γ_g which is a natural hypothesis once we want to ensure that $\widehat{R}(\lambda)$ has no zeros in Γ_b. More precisely, when solving the minimal ZDP three situations (or a mixture of them) may occur in the resulting $\widehat{R}(\lambda)$: (1) the sum of the left minimal indices is increased with respect to the original $R(\lambda)$; (2) the cancelled zeroes are replaced by some zeroes of $R_\ell(\lambda)$ (which we want to be placed in Γ_g) while the sum of left minimal indices remains unchanged; (3) no zeroes replace the cancelled ones while the sum of left minimal indices remains unchanged. This last instance occurs when a simultaneous cancellation between zeros of $R_\ell(\lambda)$ and poles of $R(\lambda)$ takes place. All three cases are illustrated by Example 1 in Section 6.1.

The paper is organized as follows: In Section 2 we review a number of preliminary facts about the Kronecker canonical form of a matrix pencil, structural invariants and centered realizations of a rmf. Section 3 contains a decomposition of the system pencil associated with $R(\lambda)$ that is the essential tool in writing down in Section 4 the class of solutions to the ZDP. Section 5 contains a characterization of the solutions to the ZDP that are J-unitary either on the imaginary axis or the unit circle. We give some relevant numerical examples and algorithms for the construction of solutions in Section 6. Some remarks concluding the paper are in Section 7. Two technical lemmas and the proof of the main result are deferred to an Appendix.

2. Preliminaries

2.1. Basic notation

By $\mathbb{C}$, $\mathbb{C}_-$, $\mathbb{C}_+$, and $\mathbb{C}_0$ we denote the complex plane, the open left half plane, the open right half plane, and the imaginary axis, respectively, and let $\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$ be the closed complex plane. By $\mathbb{D}$ and $\mathbb{D}_1(0)$ we denote the open unit disk and the unit circle, respectively. $\mathbb{D}_c:=\overline{\mathbb{C}}-\overline{\mathbb{D}}$ stands for the exterior of the closed unit disk, containing infinity.

$\mathbb{C}^{m\times n}$ is the set of $m\times n$ matrices with elements in $\mathbb{C}$. For a constant matrix A with elements in $\mathbb{C}$ we denote by A^* its conjugate transpose, and by A^T its transpose. If A is invertible A^{-*} is its conjugate transpose inverse. A Hermitian matrix A satisfies $A=A^*$, and we denote by $A>0$ if it is in addition positive definite. Let J be a constant $p\times p$ signature matrix, i.e., $J=\operatorname{diag}(I_{p_1},-I_{p_2})$, where $p_1+p_2=p$. We say A is unitary (J-unitary) if $A^*A=I$ ($A^*JA=J$). A matrix has full column (row) rank if its rank equals the number of columns (rows). By $\star$ we denote irrelevant matrix entries. I_n will stand for the identity matrix of size $n\times n$. An $m\times n$ matrix with all elements 0 will be denoted by $0_{m\times n}$, and we skip the dimensions whenever they are irrelevant. The dimension of the vectorspace $\mathcal{V}$ is denoted by $\dim(\mathcal{V})$.

2.2. Matrix pencils

We review a few basic notions about matrix pencils (see Chapter 12 in [12]).

Let A and E be $m\times n$ matrices with elements in $\mathbb{C}$. The matrix polynomial $A-\lambda E$ is called a *matrix pencil* or, briefly, *pencil*. The pencil is called *regular* if it is square ($m=n$) and has a non-vanishing determinant, i.e., $\det(A-\lambda E)\not\equiv 0$. A pencil which is not regular is called *singular*. The *normal rank* of the pencil – denoted $\operatorname{rank}_n(A-\lambda E)$ – is defined as the rank of $A-\lambda E$ for almost all $\lambda\in\mathbb{C}$ (but a finite number of points). For a regular pencil $A-\lambda E$ we have $m=n=\operatorname{rank}_n(A-\lambda E)$.

Two matrix pencils $A-\lambda E$ and $\widetilde{A}-\lambda\widetilde{E}$, with $A,E,\widetilde{A},\widetilde{E}\in\mathbb{C}^{m\times n}$, are called *strictly equivalent* if there are two constant invertible matrices $Q\in\mathbb{C}^{m\times m}$, $Z\in\mathbb{C}^{n\times n}$, such that

$$Q(A-\lambda E)Z=\widetilde{A}-\lambda\widetilde{E}. \tag{2}$$

Relation (2) induces a canonical form – the *Kronecker canonical form* – on the set of $m\times n$ pencils,

$$A_{KR}-\lambda E_{KR}:=\operatorname{diag}\left(L_{\epsilon_1},\ldots,L_{\epsilon_{\nu_r}},I_{n_\infty}-\lambda E_\infty,A_f-\lambda I_{n_f},L^T_{\eta_1},\ldots L^T_{\eta_{\nu_\ell}}\right). \tag{3}$$

Here L_k $(k \geqslant 0)$ denotes the bidiagonal $k \times (k+1)$ pencil

$$L_k := \begin{bmatrix} \lambda & & -1 & & \\ & \ddots & & \ddots & \\ & & \lambda & & -1 \end{bmatrix},$$

A_f and E_∞ are in the Jordan canonical form, with E_∞ nilpotent.

The *regular part* of $A - \lambda E$ is defined by the regular pencil $\operatorname{diag}(I_{n_\infty}, A_f) - \lambda \operatorname{diag}(E_\infty, I_{n_f})$. The finite generalized eigenvalues of $A - \lambda E$ are the eigenvalues of A_f, and their multiplicities (partial, algebraic and geometric) are defined in the usual way based on the Jordan form of A_f (see Chapters 1 and 3 in [17]). $A - \lambda E$ has an infinite generalized eigenvalue if $n_\infty > 0$, and its multiplicities (partial, algebraic and geometric) are defined as the respective multiplicities of the 0 eigenvalue of the nilpotent matrix E_∞. The union of finite and infinite generalized eigenvalues of a (possibly singular) pencil $A - \lambda E$ is denoted by $\Lambda(A - \lambda E)$ and is called the *spectrum of the pencil*.

The *singular part* of the pencil is defined by the right and left singular structure as follows. The $\epsilon_i \times (\epsilon_i + 1)$ blocks L_{ϵ_i}, $(i = 1, \ldots, \nu_r)$, are the right elementary Kronecker blocks, and $\epsilon_i \geqslant 0$ are called the *right minimal indices*. The $(\eta_j + 1) \times \eta_j$ blocks ${L_{\eta_j}}^T$, $(j = 1, \ldots, \nu_\ell)$, are the left elementary Kronecker blocks, and $\eta_j \geqslant 0$ are called the *left minimal indices*. Notice that ϵ_i and η_j can be 0.

Although our existence conditions and constructive solutions depend heavily on the Kronecker canonical form of the system pencil associated with a realization of $R(\lambda)$, we are able to express them equivalently in terms of a certain decomposition that can be achieved by using solely unitary transformations gaining therefore benefits in terms of the numerical reliability of the overall algorithm. The particular decomposition we will use may be obtained by using the Kronecker-like form of an arbitrary (possibly singular) pencil which replaces the Kronecker canonical form which is a poor numerical tool. The Kronecker-like form displays the same information as the canonical form. Precisely, any matrix pencil $A - \lambda E$, with $A, E \in \mathbb{C}^{m \times n}$, can always be reduced by unitary transformations $Q \in \mathbb{C}^{m \times m}$, $Z \in \mathbb{C}^{n \times n}$, to the block upper triangular form, called *Kronecker-like form* (see Proposition 4.7 in [33] and for numerical refinements [3,24]),

$$Q(A - \lambda E)Z = A_K - \lambda E_K := \begin{bmatrix} A_\epsilon - \lambda E_\epsilon & \star & \star & \star \\ 0 & A_\infty - \lambda E_\infty & \star & \star \\ 0 & 0 & A_f - \lambda E_f & \star \\ 0 & 0 & 0 & A_\eta - \lambda E_\eta \end{bmatrix}. \tag{4}$$

The regular part of the pencil is determined by $A_f - \lambda E_f$ and $A_\infty - \lambda E_\infty$ which are square and regular, and contain the finite and infinite generalized eigenvalues, respectively, E_f and A_∞ are invertible, and E_∞ is nilpotent.

The singular part of the pencil is determined by $A_\epsilon - \lambda E_\epsilon$ which contains the right minimal indices and has full row rank for all $\lambda \in \mathbb{C}$, and E_ϵ has full row rank, and by $A_\eta - \lambda E_\eta$ which contains the left minimal indices and has full column rank for all $\lambda \in \mathbb{C}$, and E_η has full column rank.

The following lemma which synthesizes the results of Lemmas A.2 and A.3 in [35] will play an important role in the technical machinery of the proofs.

Lemma 2. *Assume the pencils $A - \lambda E$ and $B - \lambda F$ are left and right invertible (for some λ), respectively, and $\Lambda(A - \lambda E) \cap \Lambda(B - \lambda F) = \emptyset$.*

(I) *The equation*

$$X(A - \lambda E) - (B - \lambda F)Y = C - \lambda G \tag{5}$$

always has a solution X, Y. Moreover, if the pencils $A - \lambda E$ and $B - \lambda F$ are regular then the solution is unique.

(II) *The equation*

$$(A - \lambda E)X - Y(B - \lambda F) = 0 \tag{6}$$

has the unique solution $X = 0$, $Y = 0$.

2.3. Rational matrices

We give now a short overview of some of the structural invariants of a general rmf: poles, zeros, their multiplicities, and minimal indices (see [23,8,10]).

The *normal rank* of $R(\lambda)$ – denoted $\mathrm{rank}_n(R(\lambda))$ – is the rank of the matrix $R(\lambda)$ for almost all $\lambda \in \mathbb{C}$ (but a finite number of points)

Theorem 3. *Let $R(\lambda)$ be a $p \times m$ rmf and $\lambda_0 \in \mathbb{C}$. Then there exist two square rmf's $U(\lambda)$ and $V(\lambda)$, analytic and invertible at λ_0, such that*

$$\begin{aligned} R(\lambda) &= U(\lambda)\widetilde{R}(\lambda)V(\lambda), \quad \widetilde{R}(\lambda) = \begin{bmatrix} D(\lambda) & 0_{r\times(m-r)} \\ 0_{(p-r)\times r} & 0_{(p-r)\times(m-r)} \end{bmatrix}, \\ D(\lambda) &= \mathrm{diag}\left\{(\lambda-\lambda_0)^{k_1}, (\lambda-\lambda_0)^{k_2}, \ldots, (\lambda-\lambda_0)^{k_r}\right\}, \end{aligned} \tag{7}$$

and $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_r$ are integers called the indices of the local Smith–McMillan form at λ_0. The rmf $\widetilde{R}(\lambda)$ is called the local Smith–McMillan form at λ_0, and is unique.

A point $\lambda_0 \in \mathbb{C}$ is called a *pole* (*zero*) of $R(\lambda)$ if at least one of the indices k_i in (7) is strictly negative (strictly positive). In this case the set of absolute values of the negative k_i's (the set of positive k_i's) are the *partial pole (zero) multiplicities* of $R(\lambda)$ at λ_0. The *total pole* (*zero*) *multiplicity* of $R(\lambda)$ at λ_0 is the sum of the partial pole (zero) multiplicities. By definition, $\lambda = \infty$ is a pole (zero) of $R(\lambda)$ provided $\lambda = 0$ is a pole (zero) of $R\left(\frac{1}{\lambda}\right)$, and its partial and total pole (zero) multiplicities are the partial and total pole (zero) multiplicities at $\lambda = 0$ of $R\left(\frac{1}{\lambda}\right)$. A rmf is called proper if $\lambda = \infty$ is not one of its poles; otherwise it is called improper. The McMillan degree of $R(\lambda)$ is the sum of the total multiplicities of all poles (finite and infinite).

Let $\mathbb{C}^n(\lambda)$ be the vectorspace of n-tuples over the field of rational functions in λ with coefficients in $\mathbb{C}$. Define the *degree of a polynomial vector* as the largest power of λ occuring in its components. Each vectorspace has a *minimal polynomial basis*, i.e., a polynomial basis whose sum of the degrees of its elements is minimal. The degrees of a minimal polynomial basis of the right (left) null space of $R(\lambda)$ are called the *right* (*left*) *minimal indices* of $R(\lambda)$.

Let $J = \mathrm{diag}(I_{p_1}, -I_{p_2})$ be a constant $p \times p$ signature matrix, where $p_1 + p_2 = p$. We say that a square rmf $R_\ell(\lambda)$ is J-unitary on $\mathbb{C}_0$ if $R_\ell(\lambda)^* J R_\ell(\lambda) = J$ for all $\lambda \in \mathbb{C}_0$ which are not poles of $R_\ell(\lambda)$. If, in addition, $R_\ell(\lambda)^* J R_\ell(\lambda) \leqslant J$ for every point in $\mathbb{C}_+$ which is not a pole of $R_\ell(\lambda)$, then $R_\ell(\lambda)$ is called J-inner with respect to $\mathbb{C}_+$.

We say that a square rmf $R_\ell(\lambda)$ is J-unitary on the unit circle if $R_\ell(\lambda)^* J R_\ell(\lambda) = J$ for all $\lambda \in \mathbb{D}_1(0)$ which are not poles of $R_\ell(\lambda)$. If, in addition, $R_\ell(\lambda)^* J R_\ell(\lambda) \leqslant J$ for every point in $\mathbb{D}_c$ which is not a pole of $R_\ell(\lambda)$, then $R_\ell(\lambda)$ is called J-inner with respect to $\mathbb{D}_c$.

2.4. Realization theory for rational matrices

We give here a couple of definitions and results from the general realization theory of rmf's. In order to accommodate polynomial and improper rmf's we need a special type of realization – called *centered* – that is a particular case of the general realization investigated in [1]. Centered realizations have been used to solve various singular problems formulated for rational matrix functions which may have poles at infinity (see for example [13,29,25]).

To define a centered realization fix first a $\lambda_0 \in \overline{\mathbb{C}}$ and further α, β such that

$$\begin{cases} \alpha = 1, \quad \beta = 0, \quad \text{if } \lambda_0 = \infty, \\ \alpha = \lambda_0, \quad \beta = 1, \quad \text{if } \lambda_0 \in \mathbb{C}. \end{cases} \tag{8}$$

A realization centered at λ_0 of the $p \times m$ rmf $R(\lambda)$ is a representation

$$R(\lambda) = D + (\alpha - \beta\lambda)C(\lambda E - A)^{-1}B =: \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]_{\lambda_0}, \tag{9}$$

where $A - \lambda E$ is a regular pencil, $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{p\times m}$. Whenever we use realizations centered at λ_0 we assume the implicit choice of α and β according to (8). In particular, if $\lambda_0 = \infty$ we simply drop the index λ_0 from the notation in the right-hand side of (9) and get the usual standard notation of a realization (implicitly centered at infinity). The positive integer n is called the *order* of the realization (9). The realization is called *minimal* if its order is as small as possible.

With any realization (9) we associate two pencils that play an important role in the sequel: the *pole pencil* $\mathcal{P}_R(\lambda) = A - \lambda E$ and the *system pencil*

$$\mathcal{S}_R(\lambda) = \begin{bmatrix} A-\lambda E & (\alpha-\beta\lambda)B \\ C & D \end{bmatrix} = \begin{bmatrix} A & \alpha B \\ C & D \end{bmatrix} - \lambda \begin{bmatrix} E & \beta B \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Although for every fixed λ_0 a rational matrix may be represented as in (9), if λ_0 is a pole of $R(\lambda)$ this realization has a couple of drawbacks for the problems under investigation. For example, the order of a minimal realization is strictly greater than the McMillan degree of $R(\lambda)$, D does not represent the value of $R(\lambda)$ at λ_0, and even if $R(\lambda)$ is invertible as a rmf one can not easily write down a realization for $R(\lambda)^{-1}$. To avoid these shortcomings, we introduce the notion of *proper realization*, which is a realization (9) for which $\alpha E - \beta A$ is invertible. Notice that $R(\lambda)$ has a proper realization centered at λ_0 only if it has no poles at λ_0. If the realization (9) is proper then $D = R(\lambda_0)$. In addition, if λ_0 is neither a pole nor a zero of $R(\lambda)$ then $\operatorname{rank}(D) = \operatorname{rank}_n(R(\lambda))$, $R(\lambda)$ is invertible if and only if D is invertible, and the realization for the inverse

$$R^{-1}(\lambda) = \left[\begin{array}{c|c} A-\alpha BD^{-1}C-\lambda(E-\beta BD^{-1}C) & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right]_{\lambda_0} \tag{11}$$

is proper as well. Provided the realization (9) is minimal it follows that (11) is minimal as well. However, the most important property of proper realizations (9) is that their minimal order coincides with the McMillan degree of $R(\lambda)$.

A realization (9) (or the pair $(A - \lambda E, B)$) *is controllable* at $\lambda \in \mathbb{C}$ if

$$\operatorname{rank}\begin{bmatrix} A-\lambda E & B \end{bmatrix} = n \tag{12}$$

and *is controllable* at ∞ if

$$\operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} = n. \tag{13}$$

Analogously, a realization (9) is observable (or the pair $(C, A - \lambda E)$ is observable) at a certain $\lambda \in \overline{\mathbb{C}}$ provided the pair $(A^* - \lambda E^*, C^*)$ is controllable at λ. A realization (or a pair) is called simply controllable (observable) provided it is controllable (observable) $\forall\lambda \in \overline{\mathbb{C}}$. A realization that is both controllable and observable is called *irreducible*.

The next theorem shows that for an irreducible realization there is a one-to-one correspondence between the structural elements of the rmf (poles, zeros, their partial multiplicities, and minimal indices) and the Kronecker form of the associated pole and system pencils (see Theorems 1 and 2 in [36]).

Theorem 4. *Let $R(\lambda)$ be given by an irreducible realization* (9) *of order n. Then we have*:

(1) $\operatorname{rank}_n R(\lambda) = \operatorname{rank}_n \mathcal{S}_R(\lambda) - n$, *and the right* (*left*) *minimal indices of $R(\lambda)$ are pairwise equal to the right* (*left*) *Kronecker indices of $\mathcal{S}_R(\lambda)$*.
(2) *If $\mu \in \overline{\mathbb{C}}$ is a pole* (*zero*) *of $R(\lambda)$ with partial multiplicities $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_g$, then μ is a generalized eigenvalue of $\mathcal{P}_R(\lambda)$ ($\mathcal{S}_R(\lambda)$) with partial multiplicities $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_h$, where*

$$\begin{cases} g = h, \quad \text{and} \quad k_i = s_i, \quad i = 1, \ldots, g, \quad \text{if } \mu \neq \lambda_0, \\ g \leqslant h, \quad \text{and} \quad k_i = s_i - 1, \quad i = 1, \ldots, g, \quad \text{if } \mu = \lambda_0. \end{cases}$$

In particular, a proper realization is irreducible if and only if it is minimal.

Two realizations of the same rmf

$$R(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]_{\lambda_0} = \left[\begin{array}{c|c} \widetilde{A} - \lambda \widetilde{E} & \widetilde{B} \\ \hline \widetilde{C} & \widetilde{D} \end{array}\right]_{\lambda_0} \tag{14}$$

are called equivalent if they have the same order n and there are two invertible matrices $Q \in \mathbb{C}^{n\times n}$, $Z \in \mathbb{C}^{n\times n}$, such that

$$\widetilde{A} = QAZ, \quad \widetilde{E} = QEZ, \quad \widetilde{B} = QB, \quad \widetilde{C} = CZ. \tag{15}$$

The matrices Q and Z define an *equivalence transformation*.

For our main results we need the following solution to the generalized eigenvalue assignment problem (Lemma 4.1 in [25]).

Lemma 5. *Let $(A - \lambda E, B)$ be a controllable pair, with $A, E \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, let $\Gamma \subset \overline{\mathbb{C}}$ be a set of n elements (not necessarily distinct, and possibly containing infinity), and let $\alpha, \beta \in \mathbb{C}$, not both zero, such that $\frac{\alpha}{\beta} \notin \Lambda(A - \lambda E)$ and $\frac{\alpha}{\beta} \notin \Gamma$. Then there exists a matrix $F \in \mathbb{C}^{m\times n}$ such that*

$$\Lambda(A - \lambda E + (\alpha - \beta\lambda)BF) = \Gamma. \tag{16}$$

3. Spectral decomposition of the system pencil

We give a decomposition of the system pencil associated with $R(\lambda)$ which can be computed solely by constant unitary transformations. This preliminary decomposition is the key to our main results that are given in the next section.

Throughout the rest of this paper we assume $R(\lambda)$ is given by an irreducible realization centered at infinity, either proper or not, of the form

$$R(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_1 - \lambda E_1 & A_{12} - \lambda E_{12} & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \tag{17}$$

where $\begin{bmatrix} E_1 & E_{12} \end{bmatrix}$ has full row rank and A_2 is invertible.

The reason for starting with a realization centered at infinity (and not elsewhere) lies in the availability of numerical reliable procedures and software for obtaining and manipulating such realizations [5]. However, a realization centered at λ_0 (proper or not) can be converted into a realization centered at a different point by using some simple formulas (see [29,26]). Starting from an arbitrary realization we can always obtain (17) by equivalence transformations to the left and right with unitary matrices (see [25]).

The next theorem shows the unitary decomposition while its proof may be used as a constructive basis for a numerically reliable implementation.

Theorem 6. *Let $R(\lambda)$ be a $p \times m$ rmf of McMillan degree n, normal rank r, having the sum of the left minimal indices n_ℓ, given by an irreducible realization (17) (centered at infinity) of order k, with $e := \operatorname{rank}(E) = \operatorname{rank}\begin{bmatrix} E_1 & E_{12} \end{bmatrix}$. Let $\overline{\mathbb{C}} = \Gamma_b \cup \Gamma_g$ be a disjoint partition and n_b the number of zeros of $R(\lambda)$ in Γ_b. Then there is always an update (by a left unitary equivalence transformation U) of the realization (17) and a unitary matrix Z such that*

$$\begin{bmatrix} U & 0 \\ 0 & I_p \end{bmatrix} \left[\begin{array}{cc} A - \lambda E & B \\ \hdashline C & D \end{array}\right] Z$$

$$= \left[\begin{array}{ccccc} A_{rg} - \lambda E_{rg} & B_1 - \lambda F_1 & B_2 - \lambda F_2 & B_3 - \lambda F_3 & B_4 - \lambda F_4 \\ 0 & A_b - \lambda E_b & A_{b\ell} - \lambda E_{b\ell} & B_b - \lambda F_b & B_{bn} - \lambda F_{bn} \\ 0 & 0 & A_\ell - \lambda E_\ell & B_\ell - \lambda F_\ell & B_{\ell n} - \lambda F_{\ell n} \\ 0 & 0 & 0 & 0 & B_n \\ \hdashline 0 & 0 & 0 & D_\ell & D_n \end{array}\right] \begin{array}{l} \}e - n_\ell - n_b \\ \}n_b \\ \}n_\ell \\ \}k - e \\ \}p \end{array} \tag{18}$$

where

(I) $A_{rg} - \lambda E_{rg}$ *has full row rank for all* $\lambda \in \Gamma_b$, $A_b - \lambda E_b$ *is regular with* $\Lambda(A_b - \lambda E_b) \subset \Gamma_b$, *and* B_n *is invertible*;

(II) $\operatorname{rank}\begin{bmatrix} A_\ell - \lambda E_\ell & B_\ell - \lambda F_\ell \end{bmatrix} = n_\ell, \forall \lambda \in \mathbb{C}$, $\operatorname{rank}\begin{bmatrix} E_\ell & F_\ell \end{bmatrix} = n_\ell$;

(III) $D_\ell, \begin{bmatrix} E_\ell & F_\ell \\ 0 & D_\ell \end{bmatrix}, \begin{bmatrix} A_\ell - \lambda E_\ell & B_\ell - \lambda F_\ell \\ 0 & D_\ell \end{bmatrix}$, *have all full column rank* $\forall \lambda \in \mathbb{C}$.

The matrices U and Z can be constructed by an algorithm presented below.

Proof. We give here a constructive proof which serves simultaneously as a computational procedure to determine the appropriate constant unitary matrices U and Z. Let

$$\mathcal{S}_R(\lambda) := \begin{array}{c} \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} \begin{array}{l} \}k \\ \}p \end{array} \\ \underbrace{\qquad\quad}_{k} \;\; \underbrace{\quad}_{m} \end{array} = \begin{bmatrix} A_1 - \lambda E_1 & A_{12} - \lambda E_{12} & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} \begin{array}{l} \}e \\ \}k-e \\ \}p \end{array}$$

be the system pencil associated with the realization (17), assumed to be irreducible. Hence, e is precisely the McMillan degree of $R(\lambda)$ (see Theorem 4) and (12) and (13) both hold.

Step 1. Compute a unitary Z_1 such that $\begin{bmatrix} A_2 & B_2 \end{bmatrix} Z_1 = \begin{bmatrix} 0 & B_n \end{bmatrix}$, where B_n is invertible. Define

$$\mathcal{S}_1(\lambda) := \mathcal{S}_R(\lambda)\operatorname{diag}(I_e, Z_1) = \left[\begin{array}{cc} A_1^{(1)} - \lambda E_1^{(1)} & \star \\ 0 & B_n \\ \hdashline C^{(1)} & \star \end{array}\right] \begin{array}{l} \}e \\ \}k-e. \\ \}p \end{array}$$

Step 2. Compute unitary Z_2 such that $C^{(1)} Z_2 = \begin{bmatrix} 0 & D_\ell \end{bmatrix}$ where D_ℓ has full column rank and define

$$\mathcal{S}_2(\lambda) := \mathcal{S}_1(\lambda)\operatorname{diag}(Z_2, I_{k-e}) = \left[\begin{array}{ccc} A_{11}^{(2)} - \lambda E_{11}^{(2)} & A_{12}^{(2)} - \lambda E_{12}^{(2)} & \star \\ 0 & 0 & B_n \\ \hdashline 0 & D_\ell & \star \end{array}\right] \begin{array}{l} \}e \\ \}k-e. \\ \}p \end{array}$$

From (12) and (13) we have

$$\operatorname{rank}\begin{bmatrix} A_{11}^{(2)} - \lambda E_{11}^{(2)} & A_{12}^{(2)} - \lambda E_{12}^{(2)} \end{bmatrix} = e, \quad \forall \lambda \in \mathbb{C}, \tag{19a}$$

$$\operatorname{rank}\begin{bmatrix} E_{11}^{(2)} & E_{12}^{(2)} \end{bmatrix} = e. \tag{19b}$$

Since the original realization (17) is irreducible, the zeros of $R(\lambda)$ coincide with the generalized eigenvalues (finite and infinite, multiplicities counted) of the matrix pencil $A_{11}^{(2)} - \lambda E_{11}^{(2)}$ (see Theorem 4).

Step 3. Compute unitary U_3 and Z_3 to reduce the subpencil $A_{11}^{(2)} - \lambda E_{11}^{(2)}$ to a Kronecker-like form (see (4)),

$$U_3\left(A_{11}^{(2)} - \lambda E_{11}^{(2)}\right) Z_3 = \begin{bmatrix} A_{zg} - \lambda E_{rg} & B_1 - \lambda F_1 & B_2 - \lambda F_2 \\ 0 & A_b - \lambda E_b & A_{b\ell} - \lambda E_{b\ell} \\ 0 & 0 & A_\ell - \lambda E_\ell \end{bmatrix} \begin{array}{l} \}e - n_\ell - n_b \\ \}n_b \\ \}n_\ell \end{array}, \tag{20}$$

where $A_{rg} - \lambda E_{rg}$ contains the right singular structure and the generalized eigenvalues in Γ_g, $A_b - \lambda E_b$ contains the generalized eigenvalues in Γ_b, and $A_\ell - \lambda E_\ell$ contains the left singular structure of the pencil $A_{11}^{(2)} - \lambda E_{11}^{(2)}$. Clearly, the resulting $A_{rg} - \lambda E_{rg}$ satisfies (I) in the statement. Define

$$\begin{aligned}\mathcal{S}_3(\lambda) &:= \mathrm{diag}\,(U_3, I_{k+p-e})\mathcal{S}_2(\lambda)\mathrm{diag}\,(Z_3, I_{r+k-e}) \\ &= \begin{array}{c}\left[\begin{array}{ccccc} A_{rg}-\lambda E_{rg} & B_1-\lambda F_1 & B_2-\lambda F_2 & B_3-\lambda F_3 & B_4-\lambda F_4 \\ 0 & A_b-\lambda E_b & A_{b\ell}-\lambda E_{b\ell} & B_b-\lambda F_b & B_{bn}-\lambda F_{bn} \\ 0 & 0 & A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell & B_{\ell n}-\lambda F_{\ell n} \\ 0 & 0 & 0 & 0 & B_n \\ \hdashline 0 & 0 & 0 & D_\ell & D_n \end{array}\right]\end{array}\begin{array}{l} \}e-n_\ell-n_b \\ \}n_b \\ \}n_\ell \\ \}k-e \\ \}p \end{array}.\end{aligned} \tag{21}$$

From (19) we get

$$\mathrm{rank}\,[A_\ell-\lambda E_\ell \quad B_\ell-\lambda F_\ell] = n_\ell, \quad \forall\lambda \in \mathbb{C}, \quad \mathrm{rank}\,[E_\ell \quad F_\ell] = n_\ell. \tag{22}$$

Since D_ℓ has full column rank and $A_\ell - \lambda E_\ell$ has only left singular structure, $\begin{bmatrix} E_\ell & F_\ell \\ 0 & D_\ell \end{bmatrix}$, $\begin{bmatrix} A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell \\ 0 & D_\ell \end{bmatrix}$ both have full column rank for all $\lambda \in \mathbb{C}$.

Define $U = \mathrm{diag}(U_3, I_{k-e})$, $Z = Z_1\mathrm{diag}(Z_2, I_{k-e})\mathrm{diag}(Z_3, I_{r+k-e})$. Overall, we have determined matrices U and Z such that (18) holds and all the intervening matrices satisfy the required conditions (I)–(III). □

Remark 7. When $R(\lambda)$ is a proper rmf and $E = I$, decomposition (18) essentially remains the same but with the penultimate row and last column of the pencil in its right-hand side collapsed.

Remark 8. To streamline the presentation assume further U in (18) has been absorbed as an equivalence transformation in the realization (17).

4. The class of solutions to the ZDP

The following theorem gives a characterization of minimal solutions to the ZDP formulated for a general rmf.

Theorem 9. *Given a rmf $R(\lambda)$ and a disjoint partition $\overline{\mathbb{C}} = \Gamma_g \cup \Gamma_b$, let n_b be the number of zeros of $R(\lambda)$ in Γ_b. Assume (17) is an irreducible realization of $R(\lambda)$ and let Z be a unitary matrix as in Theorem 6 for which (18) holds (see also Remark 8).*

(I) *Let α, β be two fixed numbers in $\mathbb{C}$, not both zero, such that $\frac{\alpha}{\beta} \notin \Lambda(A_b - \lambda E_b)$. The equation with pencil coefficients*

$$[\widetilde{X}\ \widehat{X}]\begin{bmatrix} A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell \\ 0 & (\alpha-\beta\lambda)D_\ell \end{bmatrix} - (A_b-\lambda E_b)\,[\widetilde{Y}\ \widehat{Y}] + [A_{b\ell}-\lambda E_{b\ell}\ \ B_b-\lambda F_b] = 0 \tag{23}$$

has always a solution for constant matrices $\widetilde{X}, \widehat{X}, \widetilde{Y}, \widehat{Y}$. Moreover, $(A_b - \lambda E_b, \widehat{X})$ is a controllable pair.

(II) *The class of minimal solutions to the ZDP is given by*

$$R_\ell(\lambda) = D_x \left[\begin{array}{c|c} A_b-\lambda E_b & -\widehat{X} \\ \hline F_x & I \end{array}\right]_{\lambda_0}, \tag{24}$$

where $\lambda_0 = \frac{\alpha}{\beta} \in \overline{\mathbb{C}}$ (as in (8)) is neither a pole nor a zero of $R_\ell(\lambda)$, D_x is any invertible matrix with $D_x \in \mathbb{C}^{p\times p}$, $\widehat{X} \in \mathbb{C}^{n_b\times p}$ is a solution to (23) and F_x is a solution to the eigenvalue assignment problem (see Lemma 5)

$$\Lambda(A_b - \lambda E_b + (\alpha-\beta\lambda)\widehat{X}F_x) \subset \Gamma_g. \tag{25}$$

The proof is deferred to the Appendix.

5. Solutions to the ZDP featuring symmetries

We investigate now the ZDP with the additional requirement on the invertible factor $R_\ell(\lambda)$ to feature a certain symmetry. Various types of symmetries are studied: J-unitary and J-inner, either with respect to the imaginary axis or the unit circle. The variable λ will be replaced with s and z whenever we discuss symmetries with respect to the imaginary axis $\mathbb{C}_0$ and the unit circle $\mathbb{D}_1(0)$, respectively.

5.1. Symmetries with respect to the imaginary axis

To reflect this symmetry accordingly, we take throughout this section the disjoint partition $\overline{\mathbb{C}} = \Gamma_g \cup \Gamma_b$, defined by

$$\Gamma_b := \mathbb{C}_+, \qquad \Gamma_g := \overline{\mathbb{C}} - \Gamma_b, \tag{26}$$

(similar results can be derived if we set $\Gamma_b := \mathbb{C}_-, \Gamma_g := \overline{\mathbb{C}} - \Gamma_b$). Therefore, without restricting generality, we may assume that $R_\ell(s)$ has no poles at infinity and, in a minimal realization

$$R_\ell(s) := \left[\begin{array}{c|c} A_x - sE_x & B_x \\ \hline C_x & D_x \end{array}\right] \tag{27}$$

centered at infinity, D_x and E_x are both invertible. We need the following preparatory result.

Proposition 10. *Let $R_\ell(s)$ be a square and invertible rmf without poles at infinity and let (27) be a minimal realization. The following are equivalent:*

(1) *$R_\ell(s)$ is J-unitary on $\mathbb{C}_0$ (J-inner with respect to $\mathbb{C}_+$).*
(2) *$D_x^* J D_x = J$ and there is an invertible Hermitian X (with $X > 0$) such that*

$$A_x^* X E_x + E_x^* X A_x + C_x^* J C_x = 0, \tag{28a}$$
$$C_x + D_x J B_x^* X E_x = 0. \tag{28b}$$

(3) *$D_x^* J D_x = J$ and there is an invertible Hermitian Y (with $Y > 0$) such that*

$$Y E_x^{-*} A_x^* + A_x E_x^{-1} Y + B_x J B_x^* = 0, \tag{29a}$$
$$C_x + D_x J B_x^* Y^{-1} E_x = 0. \tag{29b}$$

Proof. The equivalence of (1) and (2) is a particular case ($\alpha = 1, \beta = 0$) of Theorem A1 in [25]. The equivalence of (2) and (3) is straightforward by a mere rewriting of (28a) and (28b) as (29a) and (29b), respectively, with $Y := X^{-1}$. □

Combining Proposition 10 with Theorem 9 we obtain the following result.

Theorem 11. *Given a rmf $R(s)$ and the disjoint partition $\overline{\mathbb{C}} = \Gamma_g \cup \Gamma_b$ defined by (26), let n_b be the number of zeros of $R(s)$ in Γ_b. Assume (17) is an irreducible realization of $R(s)$ and let Z be a unitary matrix as in Theorem 6 for which (18) holds (see also Remark 8).*

(I) *The following are equivalent:*

(1) *There is a minimal solution to the ZDP which is J-unitary on $\mathbb{C}_0$ (J-inner with respect to $\mathbb{C}_+$).*
(2) *There is a constant solution $\widehat{X}$ to the equation with pencil coefficients*

$$\begin{bmatrix}\widetilde{X} & \widehat{X}\end{bmatrix}\begin{bmatrix}A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell\\ 0 & D_\ell\end{bmatrix}-(A_b-\lambda E_b)\begin{bmatrix}\widetilde{Y} & \widehat{Y}\end{bmatrix}+\begin{bmatrix}A_{b\ell}-\lambda E_{b\ell} & B_b-\lambda F_b\end{bmatrix}=0 \quad (30)$$

for which the Lyapunov equation

$$YE_b^{-*}A_b^*+A_bE_b^{-1}Y+\widehat{X}J\widehat{X}^*=0 \quad (31)$$

has an invertible Hermitian solution Y (with $Y>0$).

(II) *If either statement at* (I) *holds, the class of minimal solutions to the ZDP which are J-unitary on $\mathbb{C}_0$ (J-inner with respect to $\mathbb{C}_+$) is given by*

$$R_\ell(\lambda)=D_x\left[\begin{array}{c|c}A_b-\lambda E_b & -\widehat{X}\\ \hline J\widehat{X}^*Y^{-1}E_b & I\end{array}\right], \quad (32)$$

where $D_x\in\mathbb{C}^{p\times p}$ is any constant J-unitary matrix, $\widehat{X}\in\mathbb{C}^{n_b\times p}$ fulfills (30), and Y is the unique invertible solution to (31) (with $Y>0$).

Proof. (I) (1) $\Rightarrow$ (2). Since $\Gamma_b=\mathbb{C}_+$ and the ZDP has a minimal solution, say $R_\ell(s)$, Theorem 9 shows that it should have a minimal realization of the form

$$R_\ell(\lambda)=D_x\left[\begin{array}{c|c}A_b-\lambda E_b & -\widehat{X}\\ \hline F_x & I\end{array}\right], \quad (33)$$

where $\alpha=1$, $\beta=0$, and $\lambda_0=\infty$ (this choice is possible as ∞ is neither a pole nor a zero of $R_\ell(\lambda)$), D_x is any invertible matrix with $D_x\in\mathbb{C}^{p\times p}$, $\widehat{X}\in\mathbb{C}^{n_b\times p}$ is a solution to (30), and F_x is a solution to the eigenvalue assignment problem

$$\Lambda(A_b-\lambda E_b+\widehat{X}F_x)\subset\Gamma_g. \quad (34)$$

Since E_b is invertible, (34) reduces to a standard eigenvalue assignment problem and has always a solution since $(A_b-\lambda E_b,\widehat{X})$ is a controllable pair due to (I) of Theorem 9 (see for example Theorem 2.1 in [37]).

However, $R_\ell(s)$ is in addition J-unitary, which according to (3) in Proposition 10 implies that $D_x^*JD_x=J$, (31) is fulfilled for an invertible matrix Y, and

$$D_xF_x-D_xJ\widehat{X}^*Y^{-1}E_b=0.$$

This shows that $R_\ell(s)$ has indeed the form (32), where $F_x:=J\widehat{X}^*Y^{-1}E_b$. Moreover, since the zeros of a J-unitary rmf equal the conjugated poles it follows that (34) holds true for this choice of F_x.

(I) (2) $\Rightarrow$ (1) and (II). From the hypotheses in conjunction with Theorem 9 and Proposition 10 it follows that $R_\ell(s)$ defined through (33) is indeed a J-unitary minimal solution to the ZDP. This ends the proof for the J-unitary case. For J-inner we simply add the positivity condition on the unique solution to the Lyapunov equation (31). $\square$

5.2. Symmetries with respect to the unit circle

Here we give the analogue results for the case in which the symmetry is defined with respect to the unit disk. To reflect the symmetry accordingly, we take throughout this section the disjoint partition $\overline{\mathbb{C}}=\Gamma_g\cup\Gamma_b$, defined by

$$\Gamma_g:=\overline{\mathbb{D}},\quad \Gamma_b:=\overline{\mathbb{C}}-\Gamma_g, \quad (35)$$

(similar results can be derived if we set $\Gamma_b:=\mathbb{D}$, $\Gamma_g:=\overline{\mathbb{C}}-\Gamma_b$).

Since infinity belongs to Γ_b (and should be symmetrized to 0 by the J-unitary factor) and the unit circle $\mathbb{D}_1(0)$ belongs to Γ_g, we employ for the J-unitary factor a realization centered at $\lambda_0=\frac{\alpha}{\beta}$, with $\alpha=\beta=1$, of the form

$$R_\ell(z) := \left[\begin{array}{c|c} A_x - zE_x & B_x \\ \hline C_x & D_x \end{array}\right]_1 . \tag{36}$$

Notice that D_x is invertible and provided (36) is minimal $A_x - E_x$ is invertible.

Proposition 12. *Let $R_\ell(z)$ be a square and invertible rmf without poles at* 1 *and let* (36) *be a minimal realization. The following are equivalent*:

(1) *$R_\ell(z)$ is J-unitary on $\mathbb{D}_1(0)$ (J-inner with respect to $\mathbb{D}_c$).*
(2) *$D_x^* J D_x = J$ and there is an invertible Hermitian X (with $X > 0$) such that*

$$E_x^* X E_x - A_x^* X A_x + C_x^* J C_x = 0, \tag{37a}$$
$$C_x + D_x J B_x^* X (E_x - A_x) = 0. \tag{37b}$$

(3) *$D_x^* J D_x = J$ and there is an invertible Hermitian Y (with $Y > 0$) such that*

$$Y(E_x - A_x)^{-*}(A_x + E_x)^* + (E_x + A_x)(E_x - A_x)^{-1} Y + 2B_x J B_x^* = 0, \tag{38a}$$
$$C_x + D_x J B_x^* Y^{-1}(E_x - A_x) = 0. \tag{38b}$$

Proof. The equivalence of (1) and (2) is a particular case ($\alpha = 1, \beta = 1$) of Theorem A2 in [25]. We show that (2) $\Rightarrow$ (3). Rewriting (37a) as

$$(E_x + A_x)^* X (E_x - A_x) + (E_x - A_x)^* X (E_x + A_x) + 2C_x^* J C_x = 0,$$

we get further with (37b)

$$(E_x + A_x)^* X (E_x - A_x) + (E_x - A_x)^* X (E_x + A_x) + 2(E_x - A_x)^* X B_x J B_x^* X (E_x - A_x) = 0, \tag{39}$$

where we have also used that $D_x^* J D_x = J$. Denoting $Y = X^{-1}$, (39) can be put into the form of (38a) while (38b) follows directly from (37b). Reversing the arguments we can prove also (3) $\Rightarrow$ (2) which ends the whole proof. $\square$

Combining Proposition 12 with Theorem 9 we obtain the following result.

Theorem 13. *Given a rmf $R(z)$ and the disjoint partition $\overline{\mathbb{C}} = \Gamma_g \cup \Gamma_b$ defined by* (35), *let n_b be the number of zeros of $R(z)$ in Γ_b. Assume* (17) *is an irreducible realization of $R(z)$ and let Z be a unitary matrix as in Theorem* 6 *for which* (18) *holds* (*see also Remark* 8).

(I) *The following are equivalent*:

(1) *There is a minimal solution to the ZDP which is J-unitary on $\mathbb{D}_1(0)$ (J-inner with respect to $\mathbb{D}_c$).*
(2) *There is a constant solution $\widehat{X}$ to the equation with pencil coefficients*

$$\begin{bmatrix} \widetilde{X} & \widehat{X} \end{bmatrix} \begin{bmatrix} A_\ell - \lambda E_\ell & B_\ell - \lambda F_\ell \\ 0 & (1-z)D_\ell \end{bmatrix} - (A_b - \lambda E_b)\begin{bmatrix} \widetilde{Y} & \widehat{Y} \end{bmatrix} + \begin{bmatrix} A_{b\ell} - \lambda E_{b\ell} & B_b - \lambda F_b \end{bmatrix} = 0 \tag{40}$$

for which the Lyapunov equation

$$Y(E_b - A_b)^{-*}(A_b + E_b)^* + (E_b + A_b)(E_b - A_b)^{-1} Y + 2\widehat{X} J \widehat{X}^* = 0 \tag{41}$$

has an invertible Hermitian solution Y (with $Y > 0$).

(II) *If either statement at* (I) *holds, the class of minimal solutions to the ZDP which are J-unitary on* $\mathbb{D}_1(0)$ (*J-inner with respect to* $\mathbb{D}_c$) *is given by*

$$R_\ell(\lambda) = D_x \left[\begin{array}{c|c} A_b - \lambda E_b & -\widehat{X} \\ \hline J\widehat{X}^* Y^{-1}(E_b - A_b) & I \end{array}\right]_1, \tag{42}$$

where $D_x \in \mathbb{C}^{p\times p}$ *is any constant J-unitary matrix,* $\widehat{X} \in \mathbb{C}^{n_b\times p}$ *fulfills* (40)*, and Y is the unique invertible solution to* (41) (*with* $Y > 0$).

Proof. The proof follows *mutatis mutandis* from the proof of Theorem 11. □

5.3. The full row rank case

Due to its intrinsic importance, we specialize now our main result for the case of a full row normal rank $R(\lambda)$ and recapture the results in Section 4.2 of [9]. In this case, the right term of decomposition (18) collapses into

$$\left[\begin{array}{cccc} A_{rg} - \lambda E_{rg} & B_1 - \lambda F_1 & B_3 - \lambda F_3 & B_4 - \lambda F_4 \\ 0 & A_b - \lambda E_b & B_b - \lambda F_b & B_{bn} - \lambda F_{bn} \\ 0 & 0 & 0 & B_n \\ \hdashline 0 & 0 & D_\ell & D_n \end{array}\right], \tag{43}$$

where D_ℓ is square and invertible, while Eq. (23) becomes

$$(\alpha - \beta\lambda)\widehat{X}D_\ell - (A_b - \lambda E_b)\widehat{Y} + B_b - \lambda F_b = 0. \tag{44}$$

According to (I) of Lemma 2, this equation has a unique solution $(\widehat{X}, \widehat{Y})$. In particular, Theorem 11 takes the following form.

Theorem 14. *Given a rmf* $R(s)$ *which has full row rank for almost all* $s \in \mathbb{C}$*, and the disjoint partition* $\overline{\mathbb{C}} = \Gamma_g \cup \Gamma_b$ *defined by* (26)*, let* n_b *be the number of zeros of* $R(s)$ *in* Γ_b*. Assume* (17) *is an irreducible realization of* $R(s)$ *and let Z be a unitary matrix as in Theorem 6 for which* (43) *holds* (*see also Remark 8*). *The following are equivalent*:

(1) *There is a minimal solution to the ZDP which is J-unitary on* $\mathbb{C}_0$ (*J-inner with respect to* $\mathbb{C}_+$).
(2) *The Lyapunov equation*

$$YE_b^{-*}A_b^* + A_bE_b^{-1}Y + \left(B_b - A_bE_b^{-1}F_b\right)D_\ell^{-1}JD_\ell^{-*}\left(B_b - A_bE_b^{-1}F_b\right)^* = 0 \tag{45}$$

has an invertible solution Y (*with* $Y > 0$).

(II) *If either statement at* (I) *holds, the class of minimal solutions to the ZDP which are J-unitary on* $\mathbb{C}_0$ (*J-inner with respect to* $\mathbb{C}_+$) *is given by*

$$R_\ell(s) = D_x \left[\begin{array}{c|c} A_b - sE_b & \left(B_b - A_bE_b^{-1}F_b\right)D_\ell^{-1} \\ \hline -JD_\ell^{-*}\left(B_b^* - F_b^*E_b^{-*}A_b^*\right)Y^{-1}E_b & I \end{array}\right], \tag{46}$$

where $D_x \in \mathbb{C}^{p\times p}$ *is any constant J-unitary matrix and Y is the unique invertible solution* (*with* $Y > 0$) *to Eq.* (45).

Proof. For $\alpha = 1, \beta = 0$, Eq. (44) can be solved explicitly giving $\widehat{X} = (A_bE_b^{-1}F_b - B_b)D_\ell^{-1}, \widehat{Y} = E_b^{-1}F_b$. The proof then follows by simply replacing these values in Theorem 11. □

A similar result for the symmetry defined with respect to the disk can be formulated and proved *mutatis mutandis.*

6. Numerical examples and algorithms

We illustrate the proposed approach by a simple but relevant example and comment on numerically-sound algorithms for the construction of solutions.

6.1. Numerical example

For illustrative simplicity we use non-unitary transformations. Let

$$R(\lambda)=\begin{bmatrix}\lambda^3-4\lambda^2+2\lambda+5 & \lambda^3-4\lambda^2+2\lambda+6 & 2\lambda^3-8\lambda^2+4\lambda+9\\ 2\lambda^3-7\lambda^2+4\lambda+6 & 2\lambda^3-7\lambda^2+4\lambda+8 & 4\lambda^3-14\lambda^2+8\lambda+10\\ 2\lambda^3-8\lambda^2+12\lambda-6 & 2\lambda^3-8\lambda^2+12\lambda-4 & 4\lambda^3-16\lambda^2+24\lambda-14\end{bmatrix}$$

be a polynomial matrix with a realization (centered at infinity) (17) given by

$$A-\lambda E=\begin{bmatrix}0 & -\lambda & 1 & 0\\ 1 & 0 & 0 & -\lambda\\ -2-\lambda & 1 & 0 & 2\lambda\\ 0 & 0 & 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}3 & 5 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0\\ -1 & -1 & -2\end{bmatrix},$$

$$C=\begin{bmatrix}2 & -4 & 1 & 1\\ 4 & -7 & 2 & 3\\ 12 & -8 & 2 & 6\end{bmatrix},\quad D=\begin{bmatrix}7 & 10 & 11\\ 9 & 15 & 12\\ -6 & 0 & -18\end{bmatrix}.$$

The structural elements of $R(\lambda)$ are: a pole at ∞ with total multiplicity 3, a zero at 2 with multiplicity 1, a zero at ∞ with multiplicity 1, one left minimal index equal to 1, one right minimal index equal to 0 and normal rank $r=2$. We consider successively three relevant ZDP: (1) the zero at infinity is cancelled by an invertible factor; (2) the zero at 2 is cancelled by a factor which is J-unitary on the imaginary axis; (3) the zeros at 2 and ∞ are cancelled by a factor which is J-inner with respect to $\mathbb{D}_c$.

Example 1. We cancel the zero at infinity of $R(\lambda)$ by an invertible factor $R_\ell(\lambda)$ having McMillan degree equal to 1, and illustrate all three possibilities depicted in Remark 1. With

$$U=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0\\ 1 & -4 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},\quad Z=\begin{bmatrix}0 & 2 & 0 & 0 & 0 & 1 & 0\\ 0 & 4 & 0 & 1 & 0 & 0 & 0\\ 0 & 7 & 3 & 13 & 1 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 1\\ -3 & 29 & -2 & 0 & 0 & 1 & 0\\ 1 & -10 & 0 & -2 & 0 & -1 & 0\\ 1 & -9 & 1 & 1 & 0 & 0 & 0\end{bmatrix},$$

we get the decomposition (18) in the form

$$\left[\begin{array}{c:c:c:c:c} A_{rg}-\lambda E_{rg} & B_1-\lambda F_1 & B_2-\lambda F_2 & B_3-\lambda F_3 & B_4-\lambda F_4\\ \hdashline 0 & A_b-\lambda E_b & A_{b\ell}-\lambda E_{b\ell} & B_b-\lambda F_b & B_{bn}-\lambda F_{bn}\\ \hdashline 0 & 0 & A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell & B_{\ell n}-\lambda F_{\ell n}\\ \hdashline 0 & 0 & 0 & 0 & B_n\\ \hdashline 0 & 0 & 0 & D_\ell & D_n\end{array}\right]$$

$$=\left[\begin{array}{cc:c:c:ccc:c} 0 & 2-\lambda & 0 & & 0 & 0 & 1 & -\lambda\\ \hdashline 0 & 0 & 1 & & 7-\lambda & 1 & -5 & 4\lambda\\ \hdashline 0 & 0 & 0 & & 1 & 0 & -2-\lambda & 2\lambda\\ \hdashline 0 & 0 & 0 & & 0 & 0 & 0 & 1\\ \hdashline 0 & 0 & 0 & & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & & 1 & 2 & 0 & 3\\ 0 & 0 & 0 & & 0 & 2 & 8 & 6\end{array}\right].$$

We take $\lambda_0 = 1$ ($\alpha = 1$, $\beta = 1$), $\widetilde{X} = 0$, and Eq. (23) becomes $(1-\lambda)\widehat{X}D_\ell - (A_b - \lambda E_b)\widehat{Y} + B_b - \lambda F_b = 0$, having the solution $\widehat{X} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$, $\widehat{Y} = \begin{bmatrix} 6 & 1 & -5 \end{bmatrix}$. Further, we make three different choices for F_x (see (25)):

$$F_x^{(1)} = \begin{bmatrix} -1 & -\frac{4}{3} & 0 \end{bmatrix}^T, \quad F_x^{(2)} = \begin{bmatrix} 0 & \frac{2}{3} & \frac{32}{9} \end{bmatrix}^T, \quad F_x^{(3)} = \begin{bmatrix} -1 & -2 & -2 \end{bmatrix}^T.$$

With $D_x = I$ we get from (24),

$$R_\ell^{(1)}(\lambda) = D_x \left[\begin{array}{c|c} A_b - \lambda E_b & -\widehat{X} \\ \hline F_x^{(1)} & I \end{array}\right]_{\lambda_0}$$

$$= \left[\begin{array}{c|ccc} 1 & -2 & 1 & 0 \\ \hline -1 & 1 & 0 & 0 \\ -\frac{4}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]_1 = \begin{bmatrix} 2\lambda - 1 & -(\lambda - 1) & 0 \\ \frac{8}{3}(\lambda - 1) & -\frac{1}{3}(4\lambda - 7) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_\ell^{(2)}(\lambda) = D_x \left[\begin{array}{c|c} A_b - \lambda E_b & -\widehat{X} \\ \hline F_x^{(2)} & I \end{array}\right]_{\lambda_0}$$

$$= \left[\begin{array}{c|ccc} 1 & -2 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 1 & 0 \\ \frac{32}{9} & 0 & 0 & 1 \end{array}\right]_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{4}{3}(\lambda - 1) & \frac{1}{3}(2\lambda + 1) & 0 \\ -\frac{64}{9}(\lambda - 1) & \frac{32}{9}(\lambda - 1) & 1 \end{bmatrix},$$

$$R_\ell^{(3)}(\lambda) = D_x \left[\begin{array}{c|c} A_b - \lambda E_b & -\widehat{X} \\ \hline F_x^{(3)} & I \end{array}\right]_{\lambda_0}$$

$$= \left[\begin{array}{c|ccc} 1 & -2 & 1 & 0 \\ \hline -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{array}\right]_1 = \begin{bmatrix} 2\lambda - 1 & -\lambda + 1 & 0 \\ 4\lambda - 4 & -2\lambda + 3 & 0 \\ 4\lambda - 4 & -2\lambda + 2 & 1 \end{bmatrix}.$$

$R_\ell^{(1)}$ and $R_\ell^{(2)}$ both have a zero at $-\frac{1}{2}$ with multiplicity 1 and a pole at infinity with multiplicity 1, while $R_\ell^{(3)}$ has a zero at ∞ with multiplicity 1 and a pole at infinity with multiplicity 1. A direct check shows that $R_\ell^{(1)}(\lambda)R(\lambda)$

$$= \begin{bmatrix} -3\lambda^2 + 6\lambda + 1 & -3\lambda^2 + 6\lambda + 2 & -6\lambda^2 + 12\lambda + 1 \\ \frac{2}{3}\lambda^3 - \frac{17}{3}\lambda^2 + \frac{28}{3}\lambda + \frac{2}{3} & \frac{2}{3}\lambda^3 - \frac{17}{3}\lambda^2 + \frac{28}{3}\lambda + \frac{8}{3} & \frac{4}{3}\lambda^3 - \frac{34}{3}\lambda^2 + \frac{56}{3}\lambda - \frac{2}{3} \\ 2\lambda^3 - 8\lambda^2 + 12\lambda - 6 & 2\lambda^3 - 8\lambda^2 + 12\lambda - 4 & 4\lambda^3 - 16\lambda^2 + 24\lambda - 14 \end{bmatrix}$$

has a pole at ∞ with total multiplicity 3, a zero at 2 with multiplicity 1, one left minimal index equal to 2, one right minimal index equal to 0. Hence, the zero at infinity has been cancelled resulting in an increase of the sum of minimal indices to the left (case (1) of Remark 1). Correspondingly, $R_\ell^{(2)}(\lambda)R(\lambda)$

$$= \begin{bmatrix} \lambda^3 - 4\lambda^2 + 2\lambda + 5 & \lambda^3 - 4\lambda^2 + 2\lambda + 6 & 2\lambda^3 - 8\lambda^2 + 4\lambda + 9 \\ \frac{8}{3}\lambda^3 - \frac{23}{3}\lambda^2 + \frac{4}{3}\lambda + \frac{26}{3} & \frac{8}{3}\lambda^3 - \frac{23}{3}\lambda^2 + \frac{4}{3}\lambda + \frac{32}{3} & \frac{16}{3}\lambda^3 - \frac{46}{3}\lambda^2 + \frac{8}{3}\lambda + \frac{46}{3} \\ \frac{50}{9}\lambda^3 - \frac{104}{9}\lambda^2 - \frac{20}{9}\lambda + \frac{74}{9} & \frac{50}{9}\lambda^3 - \frac{104}{9}\lambda^2 - \frac{20}{9}\lambda + \frac{92}{9} & \frac{100}{9}\lambda^3 - \frac{208}{9}\lambda^2 - \frac{40}{9}\lambda + \frac{130}{9} \end{bmatrix}$$

has a pole at ∞ with total multiplicity 3, a zero at 2 with multiplicity 1, a zero at $-\frac{1}{2}$ with multiplicity 1, one left minimal index equal to 1, and one right minimal index equal to 0. Thus, the zero at infinity has been cancelled and replaced by the zero of $R_\ell^{(2)}(\lambda)$ (case (2) of Remark 1). Finally, $R_\ell^{(3)}(\lambda)R(\lambda)$

$$= \begin{bmatrix} -3\lambda^2 + 6\lambda + 1 & -3\lambda^2 + 6\lambda + 2 & -6\lambda^2 + 12\lambda + 1 \\ -5\lambda^2 + 12\lambda - 2 & -5\lambda^2 + 12\lambda & -10\lambda^2 + 24\lambda - 6 \\ -6\lambda^2 + 20\lambda - 14 & -6\lambda^2 + 20\lambda - 12 & -12\lambda^2 + 40\lambda - 30 \end{bmatrix}$$

has a pole at ∞ with total multiplicity 2, a zero at 2 with multiplicity 1, one left minimal index equal to 1, and one right minimal index equal to 0. Hence, the zero at infinity has been cancelled, no zero has replaced the cancelled one, and the sum of left minimal indices remained unchanged because a simultaneous cancellation between the zero of $R_\ell(\lambda)$ and a pole of $R(\lambda)$ at ∞ occurred (case (3) of Remark 1).

Example 2. We cancel now the zeros in $\mathbb{C}_+$ of $R(s)$ by a factor $R_\ell(s)$ which is J-unitary on the imaginary axis, where $J = \mathrm{diag}(1, -1, -1)$. $R(s)$ has one zero in $\mathbb{C}_+$ at 2. With $U = I$ and

$$Z = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 3 & 46 & 13 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -3 & -2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & -10 & -2 & 0 & -1 & 0 \\ 1 & 1 & 4 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{47}$$

we get the decomposition (18) in the form

$$\left[\begin{array}{c:c:c:c:c} A_{rg} - sE_{rg} & B_1 - sF_1 & B_2 - sF_2 & B_3 - sF_3 & B_4 - sF_4 \\ \hdashline 0 & A_b - sE_b & A_{b\ell} - sE_{b\ell} & B_b - sF_b & B_{bn} - sF_{bn} \\ \hdashline 0 & 0 & A_\ell - sE_\ell & B_\ell - sF_\ell & B_{\ell n} - sF_{\ell n} \\ \hdashline 0 & 0 & 0 & 0 & B_n \\ \hdashline 0 & 0 & 0 & D_\ell & D_n \end{array}\right]$$

$$= \left[\begin{array}{cc:c:c:ccc:c} 0 & 1 & 21-4s & & 7-s & 1 & -1 & 0 \\ \hdashline 0 & 0 & 2-s & & 0 & 0 & 1 & -s \\ \hdashline 0 & 0 & 0 & & 1 & 0 & -2-s & 2s \\ \hdashline 0 & 0 & 0 & & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & & 0 & 2 & 8 & 6 \end{array}\right].$$

We take $\lambda_0 = \infty$ ($\alpha = 1, \beta = 0$), $\widetilde{X} = 0$, and Eq. (30) becomes $\widehat{X}D_\ell - (A_b - \lambda E_b)\widehat{Y} + B_b - \lambda F_b = 0$, having the unique solution $\widehat{X} = \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{8} \end{bmatrix}$, $\widehat{Y} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. Eq. (31) has an invertible Hermitian solution $Y = -\frac{3}{256}$ and Theorem 11 shows that there exists an invertible factor which is J-unitary on $\mathbb{C}_0$ and solves the ZDP. With $D_x = I$ we get from (32),

$$R_\ell(s) = \left[\begin{array}{c|ccc} 2-s & -\frac{1}{4} & 0 & \frac{1}{8} \\ \hline -\frac{64}{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{32}{3} & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} \frac{3s+10}{3(s-2)} & 0 & -\frac{8}{3(s-2)} \\ 0 & 1 & 0 \\ \frac{8}{3(s-2)} & 0 & \frac{3s-10}{3(s-2)} \end{bmatrix}.$$

$R_\ell(s)$ has a zero at -2 with multiplicity 1, a pole at 2 with multiplicity 1, and is J-unitary. A direct check shows that

$$R_\ell(s)R(s) = \begin{bmatrix} s^3 - 4s^2 + 2s - \frac{49}{3} & s^3 - 4s^2 + 2s - \frac{46}{3} & 2s^3 - 8s^2 + 4s - \frac{101}{3} \\ 2s^3 - 7s^2 + 4s + 6 & 2s^3 - 7s^2 + 4s + 8 & 4s^3 - 14s^2 + 8s + 10 \\ 2s^3 - 8s^2 + 12s - \frac{50}{3} & 2s^3 - 8s^2 + 12s - \frac{44}{3} & 4s^3 - 16s^2 + 24s - \frac{106}{3} \end{bmatrix}$$

has a pole at ∞ with total multiplicity 3, a zero at ∞ with multiplicity 1, one left minimal index equal to 2, and one right minimal index equal to 0. The zero in $\mathbb{C}_+$ has been cancelled by a factor which is J-unitary on $\mathbb{C}_0$.

Example 3. Here we cancel the zeros in $\mathbb{D}_c$ of the polynomial matrix $R(z)$ by a factor $R_\ell(z)$ which is J-inner with respect to $\mathbb{D}_c$, where $J = \mathrm{diag}(1, -1, -1)$. $R(z)$ has two zeros in $\mathbb{D}_c$, at 2 and ∞, both with multiplicity 1. With Z in (47), we get the same decomposition as in the previous case though with another partition

$$\left[\begin{array}{c:c:c:c:c} A_{rg}-zE_{rg} & B_1-zF_1 & B_2-zF_2 & B_3-zF_3 & B_4-zF_4 \\ \hdashline 0 & A_b-zE_b & A_{b\ell}-zE_{b\ell} & B_b-zF_b & B_{bn}-zF_{bn} \\ \hdashline 0 & 0 & A_\ell-zE_\ell & B_\ell-zF_\ell & B_{\ell n}-zF_{\ell n} \\ \hdashline 0 & 0 & 0 & 0 & B_n \\ \hdashline 0 & 0 & 0 & D_\ell & D_n \end{array}\right]$$

$$=\left[\begin{array}{c:cc:c:ccc:c} 0 & 1 & 21-4z & & 7-z & 1 & -1 & 0 \\ 0 & 0 & 2-z & & 0 & 0 & 1 & -z \\ \hdashline 0 & 0 & 0 & & 1 & 0 & -2-z & 2z \\ \hdashline 0 & 0 & 0 & & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & & 0 & 2 & 8 & 6 \end{array}\right].$$

We take $\lambda_0=1$ $(\alpha=1,\beta=1)$, $\widetilde{X}=\begin{bmatrix}0 & 0\end{bmatrix}^T$, and Eq. (40) becomes $(1-z)\widetilde{X}D_\ell-(A_b-zE_b)\widehat{Y}+B_b-zF_b=0$, having the unique solution $\widehat{X}=\begin{bmatrix}1 & -1 & \frac{1}{2}\\ -\frac{1}{4} & 0 & \frac{1}{8}\end{bmatrix}$, $\widehat{Y}=\begin{bmatrix}6 & 1 & -18\\ 0 & 0 & 1\end{bmatrix}$. Eq. (41) has a unique invertible Hermitian solution $Y=\begin{bmatrix}\frac{5}{4} & -\frac{3}{16}\\ -\frac{3}{16} & \frac{1}{64}\end{bmatrix}>0$. Hence, Theorem 13 shows that there exists an invertible factor which is J-inner with respect to $\mathbb{D}_c$ and solves the ZDP. With $D_x=I$ we get from (42),

$$R_\ell(z)=\left[\begin{array}{cc|ccc} 1 & 21-4z & -1 & 1 & -\frac{1}{2} \\ 0 & 2-z & \frac{1}{4} & 0 & -\frac{1}{8} \\ \hline -2 & -42 & 1 & 0 & 0 \\ 1 & 29 & 0 & 1 & 0 \\ -2 & -50 & 0 & 0 & 1 \end{array}\right]_1=\begin{bmatrix} \frac{4z^2-7z+2}{z-2} & 2\,(1-z) & \frac{-2(z-1)}{z-2} \\ \frac{-2z(z-1)}{z-2} & z & \frac{2(z-1)}{z-2} \\ \frac{2(2z-1)(z-1)}{z-2} & 2\,(1-z) & \frac{-2z+1}{z-2} \end{bmatrix}.$$

$R_\ell(z)$ has a zero at 0 with multiplicity 1, a zero at $\frac{1}{2}$ with multiplicity 1, a pole at 2 with multiplicity 1, a pole at ∞ with multiplicity 1, and is J-inner with respect to $\mathbb{D}_c$. A direct check shows that

$$R_\ell(z)R(z)=\begin{bmatrix} -z^3-2z^2-6z+13 & -z^3-2z^2-6z+14 & -2z^3-4z^2-12z+25 \\ 3z^3-8z^2+16z-6 & 3z^3-8z^2+16z-4 & 6z^3-16z^2+32z-14 \\ -6z^2-4z+10 & -6z^2-4z+12 & -12z^2-8z+18 \end{bmatrix}$$

has a pole at ∞ with total multiplicity 3, one left minimal index equal to 3, and one right minimal index equal to 0. The zeros in $\mathbb{D}_c$ have been cancelled by a factor which is J-inner with respect to $\mathbb{D}_c$.

6.2. Numerical issues

The key to our solution is the spectral decomposition (18) whose constructive numerical algorithm (based solely on unitary transformations) is already described in details in the proof of Theorem 6. It contains as main ingredients some rank compressions (at Steps 1 and 2) which can be performed by any rank revealing algorithm based on unitary transformations (see for example [6] and the references therein) and the computation of Kronecker-like form (20) at Step 3 which can be achieved by using any of the existing staircase algorithms [20,3,24]. Though the algorithm in [20] has higher $\mathcal{O}(n^4)$ complexity, it is recommended at this initial step because the inherent recursive rank decisions are based on the singular value decomposition which is the most reliable rank revealing algorithm.

The construction of solutions to the ZDP, both in the general and symmetric cases, requires solving the pencil Eq. (23) which can be done in a numerically-sound way by the following algorithm.

6.2.1. Algorithm for solving Eq. (23).
Without restricting generality, assume $A_\ell - \lambda E_\ell$ and D_ℓ are already in the form

$$A_\ell - \lambda E_\ell = \begin{bmatrix} A_{\ell 1} - \lambda E_{\ell 1} \\ A_{\ell 2} \end{bmatrix}, \quad D_\ell = \begin{bmatrix} D_{\ell 1} \\ 0 \end{bmatrix}, \tag{48}$$

where $E_{\ell 1}$ is square and invertible, the pair $(A_{\ell 2}, A_{\ell 1} - \lambda E_{\ell 1})$ is observable, $D_{\ell 1}$ is invertible, and

$$\Lambda(A_{\ell 1} - \lambda E_{\ell 1}) \cap \Lambda(A_b - \lambda E_b) = \emptyset. \tag{49}$$

Indeed, since $A_\ell - \lambda E_\ell$ has full column rank for all λ and E_ℓ has full column rank, the form in (48) for $A_\ell - \lambda E_\ell$ can always be achieved by a unitary transformation to the left (which we assume that has already been included in U at Step 3 in the Proof of Theorem 6). Similarly, since D_ℓ has full column rank, its form in (48) can be achieved by a unitary transformation to the left Q (which we assume is already included in the solution to the ZDP by taking $R_\ell(\lambda)Q$ instead of $R_\ell(\lambda)$ in (24)). Finally, provided (49) does not hold, we perform a preliminary equivalence transformation to the left on (18) (which again can be included in U at Step 3 in the Proof of Theorem 6) of the form diag $\left(I_{e-n_\ell}, \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}, I_{k-e}\right)$, where K is a constant matrix chosen to solve an eigenvalue assignment problem such that $\Lambda(A_{\ell 1} + KA_{\ell 2} - \lambda E_{\ell 1}) \cap \Lambda(A_b - \lambda E_b) = \emptyset$. This eigenvalue assignment has always a solution due to the observability of the pair $(A_{\ell 2}, A_{\ell 1} - \lambda E_{\ell 1})$, and can be achieved in a numerically-reliable fashion without inverting explicitly $E_{\ell 1}$ by using a particular version of Algorithm 1 from Section 6.5 in [18]. Notice that only those eigenvalues of $A_{\ell 1} - \lambda E_{\ell 1}$ contained in $\Lambda(A_b - \lambda E_b)$ need to be moved (assigned).

With (48), Eq. (23) is equivalent to

$$\widetilde{X}_1(A_{\ell 1} - \lambda E_{\ell 1}) + \widetilde{X}_2 A_{\ell 2} - (A_b - \lambda E_b)\widetilde{Y} + A_{b\ell} - \lambda E_{b\ell} = 0, \tag{50a}$$
$$\widetilde{X}(B_\ell - \lambda F_\ell) + \widehat{X}_1 D_{\ell 1}(\alpha - \beta\lambda) - (A_b - \lambda E_b)\widehat{Y} + B_b - \lambda F_b = 0, \tag{50b}$$

where $\widetilde{X} = [\widetilde{X}_1 \quad \widetilde{X}_2]$, $\widehat{X} = [\widehat{X}_1 \quad \widehat{X}_2]$ have been partitioned conformably to the partitions in (48), respectively.

Part (I) of Lemma 2 shows that Eq. (50a) in the unknowns $\widetilde{X}_1$, $\widetilde{Y}$, has a unique solution for any specified choice of $\widetilde{X}_2$. For a particular choice of $\widetilde{X}_2$, (50a) can be solved by the numerically-sound algorithm proposed in [21].

Similar reasoning shows that (50b) in the unknowns $\widehat{X}_1, \widehat{Y}$, has a unique solution for $\widetilde{X}$ found at Step 1, and can be solved by the algorithm in [21] as well. Finally, $\widehat{X}_2$ can be set to an arbitrary value.

6.2.2. Construction of solutions
In the general case (see Theorem 9) one has to solve the eigenvalue assignment problem (25). By a conformal mapping $\lambda = \frac{\alpha\widetilde{\lambda}+\beta}{\beta\widetilde{\lambda}-\alpha}$, this problem reduces to one for which we can apply Algorithm 1 from Section 6.5 in [18].

In the symmetric cases (see Theorems 11 and 14), we bring D_ℓ to the compressed form in (48) by using a J-unitary matrix Q. This is possible whenever $\operatorname{rank}(D^*JD) = \operatorname{rank}(D)$ by using the algorithm in [16]. Further, one has to solve two standard Lyapunov equations (31) and (41) for which we can apply any numerically-sound algorithm which copes with Lyapunov equations with indefinite sign free term (see for example [2,14]). However, if the resulting solution Y is singular (for the J-unitary case), or is not negative definite (for the J-inner case), one still has a certain degree of freedom which, at least in certain specific cases, can be exploited to ensure a solution with the required property. Indeed, the Eq. (31) is in this case

$$Y\widetilde{A_b}^* + \widetilde{A_b}Y - \widehat{X_1}J_1\widehat{X_1}^* - \widehat{X_2}J_2\widehat{X_2}^* = 0 \tag{51}$$

where $\widetilde{A_b} := A_b E_b^{-1}$ has all eigenvalues in $\mathbb{C}_+$, $J = \operatorname{diag}(J_1, J_2)$ has been partitioned accordingly to the partition of $\widehat{X}$, J_1 and J_2 are $n_1 \times n_1$ and $n_2 \times n_2$ signature matrices, and $\widehat{X}_2$ is a free parameter. Let also $J_2 = \mathrm{diag}(I_{n_{21}}, -I_{n_{22}})$, with $n_{21} + n_{22} = n_2$, and take $\widehat{X}_2 := [\widehat{X}_{21} \quad \widehat{X}_{22}]$ partitioned conformably with J_2. Let Y_1 be the solution to

$$Y\widetilde{A_b}^* + \widetilde{A_b}Y - \widehat{X_1}J_1\widehat{X_1}^* = 0. \tag{52}$$

If (51) has a solution Y_1 which is invertible or negative definite for the J-unitary and J-inner cases, respectively, then simply take $\widehat{X}_2 = 0$. Otherwise take $\widehat{X}_2 := [0 \quad \gamma\widehat{X}_{22}]$, where $\gamma > 0$ is a scalar to be specified further and $\widehat{X}_{22}$ is chosen such that $(\widetilde{A}_b, \widehat{X}_{22})$ is a controllable pair. This is possible whenever $\widetilde{A}_b$ has the maximum number of Jordan blocks at a certain eigenvalue (taken with respect to all eigenvalues) less or equal to n_{22}. Standard Lyapunov results (see for example [28]) show that the equation

$$Y\widetilde{A_b}^* + \widetilde{A_b}Y + \widehat{X_{22}}\widehat{X_{22}}^* = 0$$

has a negative definite solution Y_2. Fix now γ such that $Y_1 + \gamma^2 Y_2$ is invertible or negative definite for the J-unitary and J-inner cases, respectively. This is always possible due to standard results on interlacing of eigenvalues of a sum of two hermitian matrices (see for example Section 1 of [11], or [19]). In both cases, γ can be simply chosen to be larger than $\sqrt{\left|\frac{\lambda_{1\max}}{\lambda_{2\min}}\right|}$, where $\lambda_{1\max}$ is the largest positive eigenvalue of Y_1 and $\lambda_{2\min}$ is the eigenvalue with the smallest modulus of Y_2. Then it is straightforward to check that (51) has the unique solution $Y_1 + \gamma^2 Y_2$ which fulfills the required property.

7. Conclusions

We have obtained a complete characterization of the minimal McMillan degree factors that cancel the zeroes of a general rmf from an arbitrary fixed region Γ_b of the complex plane. The cancelled zeroes disappear, or are replaced by zeroes in the complement of Γ_b, or the sum of the left minimal indices is increased (or any combination of these). The results encompass the cases in which the factors are invertible only, or in addition J-unitary or J-inner, either with respect to $\mathbb{C}_+$ or $\mathbb{D}_c$.

The results are more general than the ones in [35,9] as we allow for improper (possibly polynomial) rmf's and zeroes at infinity. Comparing our results with those in [9] we notice a couple of differences even in the symmetric proper case: (1) the computation of the factor implies the solution of a Lyapunov equation while in [9] of a Riccati one (though of Bernoulli type); (2) the existence conditions in Theorem 11 are simultaneously necessary and sufficient while [9] gives conditions which are either sufficient or necessary; (3) the existence conditions and the formulas for the solutions are in terms of a unitary decomposition of the original rmf's realization while those in [9] are in terms of some preliminary constructions including pseudo-inversion.

Appendix

In this appendix we prove Theorem 9. The proof is based on a preliminary lemma that gives a useful condition for $R_\ell(\lambda)$ to be a solution to the ZDP. We present two versions of this lemma, one in terms of a proper and one in terms of an improper realization of $R(\lambda)$, the first being used solely for the proof of the second.

Lemma 15. *Let $R_\ell(\lambda)$ be a minimal solution to the ZDP for $R(\lambda)$, and*

$$R(\lambda) = \left[\begin{array}{c|c} A - \lambda E & B \\ \hline C & D \end{array}\right]_{\lambda_0}, \quad R_\ell(\lambda) = \left[\begin{array}{c|c} A_x - \lambda E_x & B_x \\ \hline C_x & D_x \end{array}\right]_{\lambda_0}, \tag{53}$$

be proper minimal realizations centered at $\lambda_0 := \frac{\alpha}{\beta}$.

There exist matrices X and Y, of appropriate dimensions, such that

$$\begin{bmatrix} X & -(\alpha - \beta\lambda)B_x \end{bmatrix} \begin{bmatrix} A - \lambda E & (\alpha - \beta\lambda)B \\ C & D \end{bmatrix} = (A_x - \lambda E_x) \begin{bmatrix} Y & 0 \end{bmatrix}. \tag{54}$$

Proof. We have

$$R_\ell(\lambda)R(\lambda) = \left[\begin{array}{cc|c} A_x - \lambda E_x & (\alpha-\beta\lambda)B_xC & B_xD \\ 0 & A-\lambda E & B \\ \hline C_x & D_xC & D_xD \end{array}\right]_{\lambda_0}. \tag{55}$$

Since $R_\ell(\lambda)$ is a minimal solution to the ZDP, all its poles should cancel with zeros of $R(\lambda)$ in the product $R_\ell(\lambda)R(\lambda)$ (see for example Lemma 4.4 and Remark 4.5 in [9]; though Lemma 4.4 is actually proved for finite zeros only, its conclusions extend to infinite zeros by simply mapping the point at infinity to zero by $z = \frac{1}{\lambda}$). However, the pair $(C_x, A_x - \lambda E_x)$ is observable in (55), and therefore $A_x - \lambda E_x$ must be uncontrollable. The controllable subspace of (55) is $\widehat{\mathcal{V}} := \inf\{\mathcal{V} \mid \dim \mathcal{W} = \dim \mathcal{V},\ \ \mathrm{Im}\widehat{B} \subset \mathcal{W},\ \ \mathcal{W} := \widehat{A}\mathcal{V} + \widehat{E}\mathcal{V}\}$, where $\widehat{A} - \lambda\widehat{E} := \begin{bmatrix} A_x - \lambda E_x & (\alpha-\beta\lambda)B_xC \\ 0 & A-\lambda E\end{bmatrix}$, $\widehat{B} := \begin{bmatrix} B_xD \\ B\end{bmatrix}$ (see Section V in [34]). Since $(A-\lambda E, B)$ is controllable, $\begin{bmatrix} V^T & I\end{bmatrix}^T$ and $\begin{bmatrix} W^T & I\end{bmatrix}^T$ span $\widehat{\mathcal{V}}$ and $\widehat{\mathcal{W}} := \widehat{A}\widehat{\mathcal{V}} + \widehat{E}\widehat{\mathcal{V}}$, respectively, for two appropriate matrices V and W. Applying the equivalence transformations $Q := \begin{bmatrix} I & W \\ 0 & I\end{bmatrix}^{-1}$ to the left and $Z := \begin{bmatrix} I & V \\ 0 & I\end{bmatrix}$ to the right of (55), we get $R_\ell(\lambda)R(\lambda)$

$$= \left[\begin{array}{cc|c} A_x - \lambda E_x & -W(A-\lambda E) + (\alpha-\beta\lambda)B_xC + (A_x-\lambda E_x)V & B_xD - WB \\ 0 & A-\lambda E & B \\ \hline C_x & D_xC + C_xV & D_xD \end{array}\right]_{\lambda_0}. \tag{56}$$

In this new coordinate system, it follows from (56) and the definition of $\widehat{\mathcal{V}}$ that $-W(A-\lambda E) + (\alpha - \beta\lambda)B_xC + (A_x - \lambda E_x)V = 0$ and $B_xD - WB = 0$, which are precisely (54) for $X := W$ and $Y := V$. $\square$

Next, we give the version of Lemma 15 for an improper realization of $R(\lambda)$.

Lemma 16. *Let $R(\lambda)$ be given by an irreducible realization (17), where A_2 is invertible and $\begin{bmatrix} E_1 & E_{12}\end{bmatrix}$ has full row rank. Let also $R_\ell(\lambda)$ be a minimal solution to the ZDP given by a minimal proper realization (53) centered at $\lambda_0 := \frac{\alpha}{\beta}$, where $\lambda_0 \in \overline{\mathbb{C}}$ is not a zero of $R_\ell(\lambda)$ and not a pole of $R(\lambda)$. Then there exist matrices $X_1, X_2, Y_1, Y_2, Y_3, Y_{12} := \begin{bmatrix} Y_1 & Y_2\end{bmatrix}$, such that*

$$\begin{aligned} &\begin{bmatrix} X_1 & (\alpha-\beta\lambda)X_2 - (\alpha-\beta\lambda)B_x\end{bmatrix}\begin{bmatrix} A_1 - \lambda E_1 & A_{12} - \lambda E_{12} & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D\end{bmatrix} \\ &\quad = (A_x - \lambda E_x)\begin{bmatrix} Y_1 & Y_2 & Y_3\end{bmatrix}, \end{aligned} \tag{57}$$

$$R_\ell(\lambda)R(\lambda) = \left[\begin{array}{c|c} A-\lambda E & B \\ \hline D_xC + C_xY_{12} & D_xD + C_xY_3\end{array}\right]. \tag{58}$$

Proof. The idea of the proof is to transform the realization (17) into a centered one for which Lemma 15 holds, while keeping track of the transformations involved. Mapping back to the original realization by reversing the involved transformations will conclude the proof.

As $\beta A_1 - \alpha E_1$ and A_2 are invertible, define $V_2 := -A_2^{-1}B_2$ and $V_1 := -\beta(\beta A_1 - \alpha E_1)^{-1}((\beta A_{12} - \alpha E_{12})V_2 + B_1)$. A direct check shows

$$\left[\begin{array}{cc|c} A_1 - \lambda E_1 & A_{12} - \lambda E_{12} & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D\end{array}\right]\left[\begin{array}{cc|c} I & 0 & V_1 \\ 0 & I & V_2 \\ \hline 0 & 0 & I\end{array}\right] = \left[\begin{array}{cc|c} A_1 - \lambda E_1 & A_{12} - \lambda E_{12} & (\alpha-\beta\lambda)\widehat{B} \\ 0 & A_2 & 0 \\ \hline C_1 & C_2 & \widehat{D}\end{array}\right] \tag{59}$$

where $\widehat{B} := -E_1(\beta A_1 - \alpha E_1)^{-1}((\beta A_{12} - \alpha E_{12})V_2 + B_1)$, $\widehat{D} := D + C_1V_1 + C_2V_2$. From the partition in (59) we see that V_1 and V_2 define a Rosenbrock strict system equivalence (see Section 3.1 of [30]) which keeps unchanged the underlying rmf $R(\lambda)$. Therefore, we get

$$R(\lambda) = \left[\begin{array}{c|c} A_1 - \lambda E_1 & \widehat{B} \\ \hline C_1 & \widehat{D}\end{array}\right]_{\lambda_0}. \tag{60}$$

It is easy to check that the realization (60) is minimal since the realization (17) is irreducible. We apply Lemma 15 to $R(\lambda)$ given by (60) and get X_1 and Y_1 such that

$$\begin{bmatrix} X_1 & -(\alpha-\beta\lambda)B_x \end{bmatrix}\begin{bmatrix} A_1-\lambda E_1 & (\alpha-\beta\lambda)\widehat{B} \\ C_1 & \widehat{D} \end{bmatrix} = (A_x-\lambda E_x)\begin{bmatrix} Y_1 & 0 \end{bmatrix}. \tag{61}$$

Since $\lambda_0=\frac{\alpha}{\beta}$ is not a pole of $R_\ell(\lambda)$ and the realization (53) of $R_\ell(\lambda)$ is minimal, we have $\lambda_0\notin\Lambda(A_x-\lambda E_x)$. Further, since A_2 is invertible the pencil $(\alpha-\beta\lambda)A_2$ is regular and $\Lambda(A_x-\lambda E_x)\cap\Lambda((\alpha-\beta\lambda)A_2)$
$=\emptyset$. By (I) of Lemma 2

$$X_1(A_{12}-\lambda E_{12})+(\alpha-\beta\lambda)X_2A_2-(\alpha-\beta\lambda)B_xC_2=(A_x-\lambda E_x)Y_2 \tag{62}$$

has a unique solution X_2, Y_2. Combining (62) and (61) we get

$$\begin{aligned}&\begin{bmatrix} X_1 & (\alpha-\beta\lambda)X_2 & -(\alpha-\beta\lambda)B_x \end{bmatrix}\begin{bmatrix} A_1-\lambda E_1 & A_{12}-\lambda E_{12} & (\alpha-\beta\lambda)\widehat{B} \\ 0 & A_2 & 0 \\ C_1 & C_2 & \widehat{D} \end{bmatrix}\\ &\quad=(A_x-\lambda E_x)\begin{bmatrix} Y_1 & Y_2 & 0 \end{bmatrix}.\end{aligned} \tag{63}$$

From (63) and (59) we obtain that (57) holds for $Y_3:=-Y_1V_1-Y_2V_2$.

Finally, we show that (58) holds. By using (53) and (17) we obtain

$$R_\ell(\lambda)R(\lambda)=\left[\begin{array}{cc|c} A_x-\lambda E_x & (\alpha-\beta\lambda)B_xC & (\alpha-\beta\lambda)B_xD \\ 0 & A-\lambda E & B \\ \hline C_x & D_xC & D_xD \end{array}\right]. \tag{64}$$

We use V_1 and V_2 to perform a Rosenbrock strict system equivalence to the right on the realization (64) (see also (59)) which keeps unchanged the underlying rmf (Theorem 3.1 in [30]). After removal of the uncontrollable part, we get (58), completing in this way the whole proof of the Lemma. □

Proof of Theorem 9. (I) Let $X=\begin{bmatrix}\widetilde{X} & \widehat{X}\end{bmatrix}$, $Y=\begin{bmatrix}\widetilde{Y} & \widehat{Y}\end{bmatrix}$, $\widehat{A}-\lambda\widehat{E}:=\begin{bmatrix} A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell \\ 0 & (\alpha-\beta\lambda)D_\ell \end{bmatrix}$, $\widehat{B}-\lambda\widehat{F}:=A_b-\lambda B_b$, $\widehat{C}-\lambda\widehat{G}:=-\begin{bmatrix} A_{b\ell}-\lambda E_{b\ell} & B_b-\lambda F_b \end{bmatrix}$. Then Eq. (23) takes precisely the form (5) with hatted coefficients. Since $A_\ell-\lambda E_\ell$ contains only left singular structure and D_ℓ is injective, the pencil $\widehat{A}-\lambda\widehat{E}$ is left invertible and $\Lambda(\widehat{A}-\lambda\widehat{E})\subset\left\{\frac{\alpha}{\beta}\right\}$, while the pencil $\widehat{B}-\lambda\widehat{F}$ is regular and $\frac{\alpha}{\beta}\notin\Lambda(\widehat{B}-\lambda\widehat{F})$. From Lemma 2 we conclude that (23) has a solution $\widetilde{X},\widehat{X},\widetilde{Y},\widehat{Y}$.

Next, we show that the pair $(A_b-\lambda E_b,\widehat{X})$ is controllable. Assume by contrary that $(A_b-\lambda E_b,\widehat{X})$ is not controllable which means that at least one of the following two conditions holds:

$$\exists\lambda_0\in\mathbb{C},\quad \operatorname{rank}\begin{bmatrix} A_b-\lambda_0E_b & \widehat{X}\end{bmatrix}<n_b,\qquad \operatorname{rank}\begin{bmatrix} E_b & \widehat{X}\end{bmatrix}<n_b. \tag{65}$$

We show that the first condition leads to a contradiction. From (65) we infer the existence of a vector $x_0\neq 0$ such that $x_0^T\begin{bmatrix} A_b-\lambda_0E_b & \widehat{X}\end{bmatrix}=0$. Evaluating (23) at λ_0 and multiplying it to the left with x_0^T we get

$$\begin{bmatrix} x_0^T & x_0^T\widetilde{X}\end{bmatrix}\begin{bmatrix} A_b-\lambda_0E_b & A_{b\ell}-\lambda_0E_{b\ell} & B_b-\lambda_0F_b \\ 0 & A_\ell-\lambda_0E_\ell & B_\ell-\lambda_0F_\ell \end{bmatrix}=0.$$

However, this contradicts (19a). Analogously, the second condition in (65) leads to a contradiction to (19b). This ends the proof of part (I).

(II) We show successively: (A) Any minimal solution to the ZDP has a realization of the form (24); (B) $R(\lambda)$ given by (24) is a solution to the ZDP.

Proof of (A). Assume $R_\ell(\lambda)$ is a minimal solution to the ZDP given by a minimal realization (53), where $\lambda_0=\frac{\alpha}{\beta}\in\overline{\mathbb{C}}$ is neither a pole nor a zero of $R_\ell(\lambda)$. In particular, D_x results invertible. To prove that (53) is of the form (24), we use decomposition (18) to depict a finer pattern in (57).

The zeros in Γ_b of $R(\lambda)$ are cancelled by corresponding poles in Γ_b of $R_\ell(\lambda)$ (see Lemma 4.4 and Remark 4.5 in [9]). Hence, $A_x-\lambda E_x$ has dimension $n_b\times n_b$.

Since $R_\ell(\lambda)$ given by (53) is a solution to the ZDP, we get from Lemma 16 that there are matrices $X_1, X_2, Y_1, Y_2, Y_3, Y_{12} := [Y_1 \quad Y_2]$, such that (57) and (58) hold. Using (18) in (57) (see also Remark 8) we get

$$\begin{aligned}&[X_{11} \quad X_{12} \quad X_{13} \quad (\alpha-\beta\lambda)X_2 \quad -(\alpha-\beta\lambda)B_x] \times \\ &\left[\begin{array}{ccccc} A_{rg}-\lambda E_{rg} & B_1-\lambda F_1 & B_2-\lambda F_2 & B_3-\lambda F_3 & B_4-\lambda F_4 \\ 0 & A_b-\lambda E_b & A_{b\ell}-\lambda E_{b\ell} & B_b-\lambda F_b & B_{bn}-\lambda F_{bn} \\ 0 & 0 & A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell & B_{\ell n}-\lambda F_{\ell n} \\ 0 & 0 & 0 & 0 & B_n \\ \hdashline 0 & 0 & 0 & D_\ell & D_n \end{array}\right] \\ &= (A_x-\lambda E_x)\,[\widetilde{Y}_1 \quad \widetilde{Y}_2 \quad \widetilde{Y}_3 \quad \widetilde{Y}_4 \quad \widetilde{Y}_5]\end{aligned} \tag{66}$$

where the matrices

$$X_1 := [X_{11} \quad X_{12} \quad X_{13}], \quad [\widetilde{Y}_1 \quad \widetilde{Y}_2 \quad \widetilde{Y}_3 \quad \widetilde{Y}_4 \quad \widetilde{Y}_5] := [Y_1 \quad Y_2 \quad Y_3]\,Z \tag{67}$$

in (66) have been partitioned conformably with (18). The final step of this part of the proof is to show

$$X_{11} = 0, \quad \widetilde{Y}_1 = 0, \quad X_{12} = \widetilde{Y}_2 = I. \tag{68}$$

Provided we have shown (68), (66) collapses into $A_x - \lambda E_x = A_b - \lambda E_b$, and

$$\begin{aligned}&[X_{13} \quad -(\alpha-\beta\lambda)B_x]\begin{bmatrix} A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell \\ 0 & D_\ell\end{bmatrix} \\ &\quad -(A_b-\lambda E_b)\,[\widetilde{Y}_3 \quad \widetilde{Y}_4] + [A_{b\ell}-\lambda E_{b\ell} \quad B_b-\lambda F_b] = 0.\end{aligned} \tag{69}$$

Comparing (69) with (23) we see that $B_x = -\widehat{X}$ indeed fulfills (23), and $C_x := F_x$ should fulfill (25) because $R_\ell(\lambda)$ has its zeros in Γ_g. Therefore, $R_\ell(\lambda)$ indeed has the required form (24).

It remained to show (68). Writing (66) componentwise we get the first equation

$$X_{11}(A_{rz} - \lambda E_{rz}) = (A_x - \lambda E_x)\widetilde{Y}_1. \tag{70}$$

Since the realization of $R_\ell(\lambda)$ in (53) is minimal, $\Lambda(A_x - \lambda E_x) \cap \Gamma_g = \emptyset$, and we get with part (II) of Lemma 2 that Eq. (70) has the unique solution $X_{11} = 0$, $\widetilde{Y}_1 = 0$.

Finally, we show $X_{12} = \widetilde{Y}_2 = I$. From (66) we have

$$X_{12}(A_b - \lambda E_b) = (A_x - \lambda E_x)\widetilde{Y}_2, \tag{71}$$

$$X_{12}(A_{b\ell} - \lambda E_{b\ell}) + X_{13}(A_\ell - \lambda E_\ell) = (A_x - \lambda E_x)\widetilde{Y}_3, \tag{72}$$

$$X_{12}(B_b - \lambda F_b) + X_{13}(B_\ell - \lambda F_\ell) - (\alpha - \beta\lambda)B_x D_\ell = (A_x - \lambda E_x)\widetilde{Y}_4, \tag{73}$$

Since $\beta A_x - \alpha E_x$ and $\beta A_b - \alpha E_b$ are invertible, we rewrite (71) as

$$\widehat{X}_{12}(\widehat{A}_b - \lambda \widehat{E}_b) = (\widehat{A}_x - \lambda \widehat{E}_x)\widehat{Y}_2, \tag{74}$$

where $\widehat{X}_{12} = (\beta A_x - \alpha E_x)^{-1}X_{12}$, $\widehat{A}_x - \widehat{E}_x = (\beta A_x - \alpha E_x)^{-1}(A_x - \lambda E_x)$, $\widehat{A}_b - \lambda\widehat{E}_b = (A_b - \lambda E_b)(\beta A_b - \alpha E_b)^{-1}$, and $\widehat{Y}_2 = \widetilde{Y}_2(\beta A_b - \alpha E_b)^{-1}$. Evaluating (74) at $\frac{\alpha}{\beta}$ if $\beta \neq 0$, or from (74) directly if $\beta = 0$, we get $\widehat{X}_{12} = \widehat{Y}_2$ and (74) rewrites as $\widehat{X}_{12}(\widehat{A}_b - \lambda\widehat{E}_b) = (\widehat{A}_x - \lambda\widehat{E}_x)\widehat{X}_{12}$.

Let $x \in \operatorname{Ker}\widehat{Y}_2 (=: \mathcal{N})$. We show further that $\mathcal{N} = \emptyset$. We have $\widehat{Y}_2(\widehat{A}_b - \lambda\widehat{E}_b)x = 0$ and it follows $\dim\mathcal{N} = \dim(\widehat{A}_b\mathcal{N} + \widehat{E}_b\mathcal{N})$ which shows that $\mathcal{N}$ is a deflating subspace of the regular pencil $\widehat{A}_b - \lambda\widehat{E}_b$ (see [31]). The system pencil associated with (58) is $\mathcal{S}_{R_\ell R}(\lambda)$

$$= \begin{bmatrix} A-\lambda E & B \\ D_xC + C_xY_{12} & D_xD + C_xY_3\end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & D_x\end{bmatrix}\begin{bmatrix} A-\lambda E & B \\ C & D\end{bmatrix} + \begin{bmatrix} 0 \\ C_x\end{bmatrix}[Y_{12} \ Y_3]. \tag{75}$$

Using (18), (66), (67), and $X_{11} = 0$, $\widetilde{Y}_1 = 0$, we get that (75) is equivalent to

$$\left[\begin{array}{ccccc} A_{rg}-\lambda E_{rg} & (B_1-\lambda F_1)(\beta A_b-\alpha E_b)^{-1} & B_2-\lambda F_2 & B_3-\lambda F_3 & B_4-\lambda F_4 \\ 0 & \widehat{A}_b-\lambda\widehat{E}_b & A_{b\ell}-\lambda E_{b\ell} & B_b-\lambda F_b & B_{bn}-\lambda F_{bn} \\ 0 & 0 & A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell & B_{\ell n}-\lambda F_{\ell n} \\ 0 & 0 & 0 & 0 & B_n \\ \hdashline 0 & C_x\widehat{Y}_2 & C_x\widetilde{Y}_3 & D_xD_\ell+C_x\widetilde{Y}_4 & D_xD_n+C_x\widetilde{Y}_5 \end{array}\right] \tag{76}$$

However, this pencil will have zeros in Γ_b unless the pair $(C_x\widehat{Y}_2, \widehat{A}_b-\lambda\widehat{E}_b)$ is observable. Observability of this pair implies $\mathcal{N}=\emptyset$ because $\mathcal{N}$ is a deflating subspace of $\widehat{A}_b-\lambda\widehat{E}_b$ included in Ker $C_x\widehat{Y}_2$ (see section V in [34]). Thus $\widehat{Y}_2$, and therefore X_{12} and $\widetilde{Y}_2$ are all invertible.

Finally, by an additional equivalence transformation on $R_\ell(\lambda)$ in (53), we get $X_{12}=\widetilde{Y}_2=I$. Thus, this part of the proof ends.

Proof of (B). Conversely, let $R_\ell(\lambda)$ be given by (24), where $\widehat{X}$ and F_x fulfill (23) and (25), respectively. We show $R_\ell(\lambda)$ is a minimal solution to the ZDP. We show first that (66) is fulfilled for certain matrices "X" and "$\widetilde{Y}$". Define $X_{11}:=0$, $X_{12}:=I$, $X_{13}:=\widetilde{X}$, $\widetilde{Y}_1:=0$, $\widetilde{Y}_2:=I$, $\widetilde{Y}_3=\widetilde{Y}$, $\widetilde{Y}_4:=\widehat{Y}$. According to part I of Lemma 2, the equation

$$X_{12}(B_{bn}-\lambda F_{bn})+X_{13}(B_{\ell n}-\lambda F_{\ell n})+(\alpha-\beta\lambda)X_2B_n+(\alpha-\beta\lambda)\widehat{X}D_n=(A_b-\lambda E_b)\widetilde{Y}_5 \tag{77}$$

has a unique solution in the unknowns $X_2, \widetilde{Y}_5$, since $\Lambda((\alpha-\beta\lambda)B_n)\cap\Lambda(A_b-\lambda E_b)=\emptyset$. This shows that (66) indeed holds true, where $A_x-\lambda E_x=A_b-\lambda E_b$ and $B_x=-\widehat{X}$.

We show now that $R_\ell(\lambda)R(\lambda)$ has no zeros in Γ_b. Since (66) holds, a realization of $R_\ell(\lambda)R(\lambda)$ is given by (58) where we get with (76) for the system pencil

$$\mathcal{S}_{R_\ell R}(\lambda)Z = \left[\begin{array}{ccccc} A_{rg}-\lambda E_{rg} & B_1-\lambda F_1 & B_2-\lambda F_2 & B_3-\lambda F_3 & B_4-\lambda F_4 \\ 0 & A_b-\lambda E_b & A_{b\ell}-\lambda E_{b\ell} & B_b-\lambda F_b & B_{bn}-\lambda F_{bn} \\ 0 & 0 & A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell & B_{\ell n}-\lambda F_{\ell n} \\ 0 & 0 & 0 & 0 & B_n \\ \hdashline 0 & C_x & C_x\widetilde{Y}_3 & D_xD_\ell+C_x\widetilde{Y}_4 & D_xD_n+C_x\widetilde{Y}_5 \end{array}\right]. \tag{78}$$

Using (23) and (77) we transform (78) into the equivalent pencil

$$\left[\begin{array}{ccccc} A_{rg}-\lambda E_{rg} & B_1-\lambda F_1 & B_2-\lambda F_2 & B_3-\lambda F_3 & B_4-\lambda F_4 \\ 0 & A_b-\lambda E_b & 0 & (\alpha-\beta\lambda)B_xD_\ell & (\alpha-\beta\lambda)B_xD_n \\ 0 & 0 & A_\ell-\lambda E_\ell & B_\ell-\lambda F_\ell & B_{\ell n}-\lambda F_{\ell n} \\ 0 & 0 & 0 & 0 & B_n \\ \hdashline 0 & C_x & 0 & D_xD_\ell & D_xD_n \end{array}\right] \tag{79}$$

which shows that the zeroes of $R_\ell(\lambda)R(\lambda)$ are among the union of $\Lambda(A_{rg}-\lambda E_{rg})$, $\Lambda(A_\ell-\lambda E_\ell)$ and the zeros of $\left[\begin{array}{c|c} A_b-\lambda E_b & B_xD_\ell \\ \hline C_x & D_xD_\ell \end{array}\right]_{\lambda_0} = R_\ell(\lambda)D_\ell$. However, $\Lambda(A_{rg}-\lambda E_{rg})\subset\Gamma_g$, $\Lambda(A_\ell-\lambda E_\ell)=\emptyset$, while the zeros of $R_\ell(\lambda)D_\ell$ are in Γ_g. Hence, $R_\ell(\lambda)R(\lambda)$ has no zeros in Γ_b, $R_\ell(\lambda)$ is indeed a solution to the ZDP, while its minimality follows from the order n_b of the realization (24). □

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